General Covariance and the Objectivity of Space-Time Point-Events: The Physical Role of Gravitational and Gauge Degrees of Freedom in General Relativity

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Abstract

This paper deals with a number of technical achievements that are instrumental for a dis-solution of the so-called *Hole Argument* in general relativity. Such achievements include:

1) the analysis of the *Hole* phenomenology in strict connection with the Hamiltonian treatment of the initial value problem. The work is carried through in metric gravity for the class of Christoudoulou-Klainermann space-times, in which the temporal evolution is ruled by the *weak* ADM energy;

2) a re-interpretation of *active* diffeomorphisms as *passive and metric-dependent* dynamical symmetries of Einstein's equations, a re-interpretation which enables to disclose their (up to now unknown) connection to gauge transformations on-shell; understanding such connection also enlightens the real content of the Hole Argument or, better, dis-solves it together with its alleged "indeterminism";

3) the utilization of the Bergmann-Komar *intrinsic pseudo-coordinates* [1], defined as suitable functionals of the Weyl curvature scalars, as tools for a peculiar gauge-fixing to the super-hamiltonian and super-momentum constraints;

4) the consequent construction of a *physical atlas* of 4-coordinate systems for the 4-dimensional *mathematical* manifold, in terms of the highly non-local degrees of freedom of the gravitational field (its four independent *Dirac observables*). Such construction embodies the *physical individuation* of the points of space-time as *point-events*, independently of the presence of matter, and associates a *non-commutative structure* to each gauge fixing or four-dimensional coordinate system;

5) a clarification of the multiple definition given by Peter Bergmann [2] of the concept of *(Bergmann) observable* in general relativity. This clarification leads to the proposal of a *main conjecture* asserting the existence of i) special Dirac's observables which are also Bergmann's observables, ii) gauge variables that are coordinate independent (namely they behave like the tetradic scalar fields of the Newman-Penrose formalism). A by-product of this achievements is the falsification of a recently advanced argument [3] asserting the absence of (any kind of) *change* in the observable quantities of general relativity.

6) a clarification of the physical role of Dirac and gauge variables as their being related to *tidal-like* and *inertial-like* effects, respectively. This clarification is mainly due to the fact that, unlike the standard formulations of the equivalence principle, the Hamiltonian formalism allows to define notion of "force" in general relativity in a natural way;

7) a proposal showing how the physical individuation of point-events could in principle be implemented as an experimental setup and protocol leading to a "standard of space-time" more or less like atomic clocks define standards of time.

We conclude that, besides being operationally essential for building measuring apparatuses for the gravitational field, the role of matter in the non-vacuum gravitational case is also that of *participating directly* in the individuation process, being involved in the determination of the Dirac observables. This circumstance leads naturally to a peculiar new kind of *structuralist* view of the general-relativistic concept of space-time, a view that embodies some elements of both the traditional *absolutist* and *relational* conceptions. In the end, space-time point-events maintain a *peculiar sort of objectivity*. Some hints following from our approach for the quantum gravity programme are also given.

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I. INTRODUCTION.

General Relativity is commonly thought to imply that space-time points have no *intrinsic physical meaning* due to the general covariance of Einstein's equations. This feature is implicitly described in standard modern textbooks by the statement that solutions to the Einstein's equations related by (*active*) diffeomorphisms have physically identical properties. Such kind of equivalence, which also embodies the modern understanding of Einstein's *Hole Argument*, has been named as *Leibniz equivalence* in the philosophical literature by Earman and Norton [4] and exploited to the effect of arguing against the *substantivalist* and in defense of the *relational* conception of space-time.

This paper is inspired by the belief that *Leibniz equivalence* is not and cannot be the last word about the *intrinsic physical properties* of space-time, well beyond the needs of the empirical grounding of the theory. Specifically, the content of the paper should be inscribed in the list of the various attempts made in the literature to gain an intrinsic *dynamical characterization of space-time points in terms of the gravitational field itself*, besides and beyond the mathematical individuation furnished to them by the coordinates. We refer in particular to old hints offered by Synge, and to the attempts successively sketched by Komar, Bergmann and Stachel. Actually, we claim that we have pursued this line of thought till its natural end.

The Hole Argument is naturally spelled out within the configurational Lagrangian framework of Einstein's theory. Its very formulation, however, necessarily involve the mathematical structure of the initial value problem which, on the other hand, is intractable within that framework. The proper way to deal with such problem is indeed the ADM Hamiltonian framework with its realm of Dirac observables and gauge variables. But then the real difficulty is just the connection between such different frameworks, particularly from the point of view of *symmetries*.

Our analysis starts off from a nearly forgotten paper by Bergmann and Komar which enables us to enlighten this correspondence of symmetries and, in particular, that existing between *active diffeomorphisms* of the configurational approach and *gauge transformations* of the Hamiltonian viewpoint. Understanding this relation also enlightens the content of the Hole Argument definitely or, better, dis-solves it together with its alleged "indeterminism".

Although we believe that the topics we discuss on the basis of the acquired knowledge about the above correspondence of symmetries are conceptually very significant, they are at the same time highly technical and complicated so that it is important to keep a firm grip on the key leading ideas, which are essentially two. Precisely:

1) The Komar-Bergmann *intrinsic pseudo-coordinates*, constructed in terms of the eigenvectors of the Weyl tensor, are exploited to introduce a peculiar gauge-fixing to the super-hamiltonian and super-momentum constraints in the canonical reduction of general relativity. The upshot is that, given an arbitrary coordinate system, the values of the Dirac observables for the vacuum (i.e., the phase space intrinsic degrees of freedom of the gravitational field), whose dependence on space and time is indexed by the chosen coordinates, *reproduces* precisely these latter as the Komar-Bergmann intrinsic coordinates in the *chosen gauge*. This means that the *physical individuation* of manifold's points into *point-events*

is realized by the *intrinsic components* (just four!) of the gravitational field, in a gauge dependent fashion.

2) The original *multiple definition* offered by Bergmann of the concept of "(Bergmann) observable" [2], a definition that contains some ambiguities, is spelled out fully. Such observables are required to be invariant under standard *passive* diffeomorphisms and uniquely *predictable* from the initial data. Once fully clarified, the concept of *predictability* entails that, in order Bergmann's *multiple definition* be consistent, only four of such observables can exist for the vacuum gravitational field, and can be nothing else than tensorial Lagrangian counterparts of the Hamiltonian Dirac observables. We formalize this result and related consequences into a *main conjecture*, which essentially amounts to claiming the internal consistency of Bergmann's *multiple definition*. Incidentally, this result helps in showing that a recent claim about the absence of any kind of *change* in general relativity is not mathematically justified.

Other achievements are consequences and refinements of these leading ideas or reflections and hints originated by their development.

Previous partial accounts of the material of this paper can be found in Refs. [5,6].

The *Hole Argument* (see below in this Introduction for details) has a long history which began in late 1913 and crossed Einstein's path repeatedly. Einstein found an answer only in late 1915 in terms of the now so-called *point-coincidence argument* which re-established his confidence in general covariance, but lead him to the final conviction that space and time must forfeit the *last remnant of physical objectivity*. In Einstein's own words [7]:

> "That the requirement of general covariance, *which takes away from space and time the last remnant of physical objectivity*, is a natural one, will be seen from the following reflexion. All our space-time verifications invariably amount to a determination of space-time coincidences. If, for example, events consisted merely in the motion of material points, then ultimately nothing would be observable but the meetings of two or more of these points. Moreover, the results of our measurings are nothing but verifications of such meetings of the material points of our measuring instruments with other material points, coincidences between the hands of a clock and points on the clock dial, and observed point-events happening at the same place at the same time. The introduction of a system of reference serves no other purpose than to facilitate the description of the totality of such coincidences".

At first sight it could seem from these words that Einstein simply equated general covariance with the unavoidable *arbitrariness of the choice of coordinates*, a fact that, in modern language, can be translated into *invariance under passive diffeomorphisms*. The essence of the *point-coincidence argument* (a terminology introduced by Stachel in 1980), which satisfied Einstein doubts at the end of 1915, seems to be well in tune with the Machian epistemology he shared at the time, in particular as regards the ontological privilege of "*bodies*" or "*fields*" versus "*space*". This argument, however, offered mainly a pragmatic solution of the issue and was based on a very idealized model of physical measurement where all possible observations reduce to the intersections of the world-lines of observers, measuring instruments, and measured physical objects. Also, it does not do full justice to the field concept which is the essence of the theory. Furthermore, this solution left unexplored some important aspects

of the role played by the metric tensor in the Hole Argument as well as of the related underlying full mathematical structure of the theory. On the other hand, that Einstein was not too much worried about such hidden properties of the metric tensor, is also shown by the circumstance that the subsequent *geometro-dynamical* re-interpretation of the metric field in more or less *substantival* terms, that took place in the late years of his life, did not lead Einstein to a re-visitation of the Hole phenomenology.

That the Hole Argument was in fact a subtler issue that Einstein seemingly thought in 1915 and that it consisted in much more than mere arbitrariness in the choice of the coordinates[1], has been revealed by a seminal talk given by John Stachel in 1980 [8], which gave new life to the original Hole Argument. Stachel's rediscovery of the Hole Argument was followed by a later important work in philosophy of science published by John Earman and John Norton in 1987 [4]. This work was directed against any possible *substantivalist* interpretation of the space-time concept of general relativity and in favor of the view that such space-time - although not *unreal* and not deprived of any *reality* at all - has *no reality independent of the bodies or fields it contains.* Earman-Norton's provocation raised a rich philosophical debate that is still alive today (the reader is referred to the works of John Norton for an extensive bibliography [9–12]). Getting involved in this debate is not our aim here: after all, we believe that this debate, which tends to reproduce, mainly *by analogy*, the classical Newton-Leibniz dispute on the alternative substantivalism/relationalism (a thorough exposition of this unending debate can be found in Ref. [13]), occasionally oversteps the philosophical latitude allowed by the very structure of general relativity. We shall resume these issues later on and offer our conclusive view in the Concluding Survey.

This said, one could legitimately ask why are we proposing to re-discuss the Hole Argument from a technical point of view, given the fact that the debate alluded above is mainly of philosophical interest and that all modern technical expositions of general relativity forget completely the Hole Argument *as such.* Thus, beside stressing the obvious circumstance that any conceptual clarification of the foundations of general relativity is welcome in the perspective of the quantum gravity programme, we owe the reader from the beginning a justification for our proposal of technical re-visitation of the issue. We shall do it, but only after having addressed a number of preliminary topics.

First of all, before briefly expounding the modern version of the Hole Argument, let us recall the basic mathematical concept that underlies it, namely the concept of *active diffeomorphism* and its consequent action on the tensor fields defined on a differentiable manifold. Our manifold will be the mathematical manifold M^4, the first layer of the would-be physical space-time of general relativity. Consider a (geometrical or *active*) diffeomorphism ϕ which maps points of M^4 to points of M^4: $\phi : p \rightarrow \; p' = \phi \cdot p$, and its tangent map ϕ^* which maps tensor fields $T \rightarrow \; \phi^* \cdot T$ in such a way that $[T](p) \rightarrow \; [\phi^* \cdot T](p) \equiv [T^{'}](p)$. Then $[\phi^* \cdot T](p) = [T](\phi^{-1} \cdot p)$. It is seen that the transformed tensor field $\phi^* \cdot T$ is a *new* tensor field whose components in general will have at p values that are *different* from those of the components of T. On the other hand, the components of $\phi^* \cdot T$ have at p' - by construction -

[1] In fact, however, Einstein's argument was not so naive, see below.

the same values that the components of the original tensor field T have at p: $T^{'}(\phi \cdot p) = T(p)$ or $T'(p) = T(\phi^{-1} \cdot p)$. The new tensor field $\phi^* \cdot T$ is called the *drag-along* of T.

For later use it is convenient to recall that there is another, non-geometrical - so-called *dual* - way of looking at the active diffeomorphisms, which, incidentally, is more or less the way in which Einstein himself formulated the original Hole Argument[2]. This *duality* is based on the circumstance that in each region of M^4 covered by two or more charts there is a one-to-one correspondence between an *active* diffeomorpshism and a specific coordinate transformation (or *passive* diffeomorphism). Note, incidentally, that such *duality* between the two types of transformations has often blurred the important conceptual distinction between *active* and *passive* diffeomorphisms, which remains true and should be kept clearly in mind. The coordinate transformation $\mathcal{T}_\phi : x(p) \rightarrow \;\; x'(p) = [\mathcal{T}_\phi x](p)$ which is *dual* to the active diffeomorphism ϕ is defined such that $[\mathcal{T}_\phi x](\phi \cdot p) = x(p)$. In its essence, this *duality* transfers the functional dependence of the new tensor field in the new coordinate system to the old system of coordinates. By analogy, the coordinates of the new system $[x']$ are said to have been *dragged-along* with the *active* diffeomorphism ϕ. A more detailed discussion or, better, the right mathematical way of looking *passively* at the *active* diffeomorphisms will be expounded in Section II, where the Bergmann-Komar [14] general group Q of *passive dynamical symmetries* of Einstein's equations will be introduced.

Now, the Hole Argument, in its modern version, runs as follows. Consider a general-relativistic space-time, as specified by the four-dimensional mathematical manifold M^4 and by a metrical tensor field g which *represents at the same time the chrono-geometrical and causal structure of space-time and the potential for the gravitational field*. The metric g is a solution of the generally-covariant Einstein equations. If any non-gravitational physical fields are present, they are represented by tensor fields that are also dynamical fields, and that appear as sources in the Einstein equations.

Assume now that M^4 contains a *Hole* $\mathcal{H}$: that is, an open region where all the non-gravitational fields are zero. On M^4 we can prescribe an *active* diffeomorphism ϕ that re-maps the points inside $\mathcal{H}$, but blends smoothly into the identity map outside $\mathcal{H}$ and on the boundary. Now, *just because Einstein's equations are generally covariant* so that they can be written down as *geometrical relations*, if g is one of their solutions, so is the *drag-along* field $g' = \phi^* \cdot g$. By construction, for any point $p \in \mathcal{H}$ we have (geometrically) $g'(\phi \cdot p) = g(p)$, but of course $g'(p) \neq g(p)$ (also geometrically). Now, what is the correct interpretation of the new field g'? Clearly, the transformation entails an *active redistribution of the metric over the points of the manifold*, so the crucial question is whether, to what extent, and how the points of the manifold are primarily *individuated*.

In the mathematical literature about topological spaces, it is always implicitly assumed that the entities of the set can be distinguished and considered separately (provided the Hausdorff conditions are satisfied), otherwise one could not even talk about point mappings or homeomorphisms. It is well known, however, that the points of a homogeneous space can-

[2]This point is very interesting from an historical point of view [10].

not have any intrinsic individuality[3]. Quite aside from the phenomenological stance implicit in Weyl's quoted words, there is only one way to individuate points at the mathematical level that we are considering: namely by *coordinatization*, a procedure that transfers the individuality of 4-tuples of real numbers to the elements of the topological set. Precisely, we introduce by convention *a standard* coordinate system for the *primary individuation* of the points (like the choice of *standards* in metrology). Then, we can get as many different *names*, for what we consider the same primary individuation, as the coordinate charts containing the point in the chosen atlas of the manifold. We can say, therefore, that all the relevant transformations operated on the manifold M^4 (including *active* diffeomorphisms which map points to points), even if viewed in purely geometrical terms, *must* be constructible in terms of coordinate transformations. In this way we cross necessarily from the domain of *geometry* to that of *algebra* (for a related viewpoint based on more elaborate mathematical structures like fibered manifolds, see Stachel [16]).

Let us go back to the effect of this *primary* mathematical individuation of manifold points. If we now think of the points of $\mathcal{H}$ as also *physically individuated* spatio-temporal events even before the metric is defined, then g and g' must be regarded as *physically distinct* solutions of the Einstein equations (after all, as already noted, $g'(p) \neq g(p)$ at the *same* point p). This, however, is a devastating conclusion for the causality, or better, *determinateness* of the theory, because it implies that, even after we completely specify a physical solution for the gravitational and non-gravitational fields outside the Hole - for example, on a Cauchy surface for the initial value problem, assuming for the sake of argument that this intuitive and qualitative wording is mathematically correct, see Section V - we are still *unable to predict uniquely the physical solution within the Hole.* Clearly, if general relativity has to make any sense as a physical theory, there must be a way out of this foundational quandary, *independently of any philosophical consideration.*[4]

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[3]As Hermann Weyl [15] puts it: "There is no distinguishing objective property by which one could tell apart one point from all others in a homogeneous space: at this level, fixation of a point is possible only by a *demonstrative act* as indicated by terms like *this* and *there*."

[4]In this paper we prefer to avoid the term *determinism*, and replace it by adopting *determinateness* or *causality*, because we believe that the metaphysical flavor of the former tends to overstate the issue at hand. This is especially true if *determinism* is taken in opposition to *indeterminism*, which is not mere absence of *determinism*. The issue of determinism was however the crucial ingredient of the quoted Earman-Norton's argument [8], which, roughly speaking, ran as follows. Under the substantivalist assumption, (manifold) space-time points possess an individual reality of their own (i.e., *independent* of bodies and fields of any kind), so that the rearrangement of the metric field against their background, as envisaged in the Hole Argument, would originate a true change in the physical state of space-time. But now - as said before - if diffeomorphically related metric fields represent different physical states, then any prescription of initial data outside the Hole would fail to determine a corresponding solution of the Einstein equations inside the Hole, because many are equally possible. In this way, Earman and Norton intend to confront the substantivalist with a dire dilemma: accept *indeterminism*, or abandon *substantivalism*. We want to add here, however, that we find it rather arbitrary to transcribe Newtonian absolutism (or at least part of

In the modern understanding, the most widely embraced escape from the (mathematical) strictures of the Hole Argument (which is essentially an update to current mathematical terms of the pragmatic solution adopted by Einstein), is to *deny that diffeomorphically related mathematical solutions represent physically distinct solutions.* With this assumption, *an entire equivalence class of diffeomorphically related mathematical solutions represents only one physical solution.* This statement, is implicitly taken as obvious in the contemporary specialized literature (see, e.g. Ref. [21]), and, as already said, has come to be called *Leibniz equivalence* in the philosophical literature.

It is seen at this point that the conceptual content of general covariance is far more deeper than the simple invariance under arbitrary changes of coordinates. Stachel [22,23] has given a very enlightening analysis of the meaning of general covariance and of its relations with the Hole Argument, expounding the conceptual consequences of the *de facto* Einstein's acceptance of modern Leibniz equivalence through the point-coincidence argument. Stachel stresses that asserting that g and $\phi^* \cdot g$ represent *one and the same gravitational field* is to imply that *the mathematical individuation of the points of the differentiable manifold by their coordinates has no physical content until a metric tensor is specified.* In particular, coordinates lose any *physical significance whatsoever* [9]. Furthermore, as Stachel emphasizes, if g and $\phi^* \cdot g$ must represent the same gravitational field, they cannot be physically distinguishable in any way. So when we act on g with an active diffeomorphisms to create the drag-along field $\phi^* \cdot g$, *no element of physical significance* can be left behind: in particular, nothing that could identify a point p of the manifold as the *same* point of space-time for both g and $\phi^* \cdot g$. Instead, when p is mapped onto $p' = \phi \cdot p$, it *brings over its identity*, as specified by $g'(p') = g(p)$. A further important point made by Stachel is that simply

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it) into the so-called *manifold substantivalism*, no less than to assert that general relativity is a *relational* theory in an allegedly Leibnizian sense. As emphasized by Rynasiewicz [17], the crucial point is that the historical debate presupposed a clear-cut distinction between *matter* and *space*, or between *content* and *container*; but by now these distinctions have been blurred by the emergence of the so-called *electromagnetic view of nature* in the late nineteenth century (for a detailed model-theoretical discussion of this point see also Friedman's book [18]). Still, although some might argue (as Earman and Norton do [4]) that the metric tensor, *qua* physical field, cannot be regarded as the *container* of other physical fields, we argue that the metric field and, in particular, its dynamical degrees of freedom have *ontological priority over all other fields.* This preeminence has various reasons [19], but the most important is that the metric field tells all of the other fields how to behave causally. We also agree with Friedman [18] that, in consonance with the general-relativistic practice of not counting the gravitational energy induced by the metric as a component of the total energy, *we should regard the manifold, endowed with its metric, as space-time, and leave the task of representing matter to the stress-energy tensor.* It is just because of this priority, beside the fact that the Hole *is* pure gravitational field, that we maintain, unlike other authors (see for example Ref. [20]), that the issue of the individuation of points of the manifold as *physical* point-events should be discussed primarily in the context of the vacuum gravitational field, without any recourse to non-gravitational entities, except perhaps at the operational level. *Nevertheless even matter plays peculiar role in the process of individuation when present.* We shall come back to such qualifications in the Concluding Survey.

because a theory has generally covariant equations, it does not follow that the points of the underlying manifold must lack any kind of physical individuation. Indeed, what really matters is that there can be no *non-dynamical individuating field* that is specified *independently* of the dynamical fields, and in particular independently of the metric. If this was the case, a *relative* drag-along of the metric with respect to the (supposedly) individuating field would be physically significant and would generate an inescapable Hole problem. Thus, the absence of any non-dynamical individuating field, as well as of any dynamical individuating field independent of the metric, is the crucial feature of the purely gravitational solutions of general relativity as well as of the very *concept* of *general covariance*.

This conclusion led Stachel to the conviction that space-time points *must* be *physically* individuated *before* space-time itself acquires a physical bearing, and that the metric itself plays the privileged role of *individuating field*: a necessarily *unique role* in the case of space-time *without matter*. More precisely, Stachel claimed that this individuating role should be implemented by four invariant functionals of the metric, already considered by Bergmann and Komar [1] (see Section IV). However, he did not follow up on his suggestion. As a matter of fact, as we shall see, the question is not straightforward.

Let us come to our program of re-visitation of the Hole Argument. There are many reasons why one should revisit the Hole Argument nowadays, quite apart from any *philosophical* interest. First of all, this paper is inspired by the conviction that a deeper mathematical clarification of the Hole Argument is needed anyway because the customary statement of Leibniz equivalence is too much synthetic and shallow. As aptly remarked by Michael Friedman [18], if we stick to simple Leibniz equivalence, "how do we describe this physical situation *intrinsically* ?". The crucial point of the Hole issue is that the mathematical representation of space-time provided by general relativity under the condition of general covariance evidently contains *superfluous structure* hidden behind Leibniz equivalence and that this structure must be isolated. At the level of general covariance, only the equivalence class is physically real so that, on this understanding, general covariance is invariably an unbroken symmetry and the physical world is to be described in a diffeomorphically invariant way. Of course, the price to be paid is that the values of all fields at manifold points as specified by the coordinates, are not physically real. One could say [24] that only the *relations* among these field values are invariant an thereby real. However, this point of view collapses if we are able to isolate the intrinsic content of Leibniz equivalence and to individuate physically the manifold's point in an independent way. On the other hand, this isolation appears to be required *de facto* both by any explicit solution of Einstein's equation, which requires specification of the arbitrariness of coordinates, and by the empirical foundation of the theory: after all any effective kind of measurement requires in fact a definite physical individuation of space-time points in terms of physically meaningful coordinates. Summarizing, it is evident that breaking general covariance is a pre-condition for the isolation of the superfluous structure hidden within *Leibniz equivalence*.

Secondly, the program of the physical individuation of space-time points sketched by John Stachel must be completed because, as it will appear evident in Section IV, the mere recourse to the four functional invariants of the metric alluded to by Stachel cannot do, by itself, the job of physically individuating space-time points. Above all, it is essential to realize from the beginning that - by its very formulation - the *Hole Argument* is *inextricably entangled* with

the initial value problem although, strangely enough, it has never been explicitly discussed in that context in a systematic way. Possibly the reason is that most authors have implicitly adopted the Lagrangian approach, where the Cauchy problem is intractable because of the non-hyperbolic nature of Einstein's equations[5] (see Ref. [28] for an updated review).

Our investigation will be based on the Hamiltonian formulation of general relativity for it is precisely the entanglement referred to above that provides the right point of attack of the Hole problem. This is no surprise, after all, since the constrained Hamiltonian approach is just the *only* proper way to analyze the initial value problem of that theory and to find the *deterministically predictable observables* of general relativity. It is not by chance that the modern treatment of the initial value problem within the Lagrangian configurational approach [28] must in fact mimic the Hamiltonian methods (see more in Section V).

Finally, only in the Hamiltonian approach can we isolate the *gauge variables*, which carry the descriptive arbitrariness of the theory, from the *Dirac observables* (DO), which are gauge invariant quantities providing a coordinatization of the reduced phase space of general relativity, and are subjected to hyperbolic (and therefore "*determinated*" or "*causal*" in the customary sense) evolution equations.

Just in the context of the Hamiltonian formalism, we find the tools for completing Stachel's suggestion and exploiting the old proposal advanced by Bergmann and Komar for an intrinsic labeling of space-time points by means of the eigenvalues of the Weyl tensor. Precisely, Bergman and Komar, in a series of papers [29,1,30] introduced suitable invariant scalar functionals of the metric and its first derivatives as *invariant pseudo-coordinates*[6]. As already anticipated, we shall show that such proposal can be utilized in constructing a peculiar *gauge-fixing to the super-hamiltonian and super-momentum constraints* in the canonical reduction of general relativity. This gauge-fixing makes the *invariant pseudo-coordinates* into effective *individuating fields* by forcing them to be *numerically* identical with ordinary coordinates: in this way the individuating fields turn the *mathematical* points of space-time into *physical point-events*. Eventually, we discover that what really individuates space-time points physically are the very *degrees of freedom of the gravitational field*. As a consequence, we advance the *ontological* claim that - physically - Einstein's vacuum space-time is literally *identified* with the autonomous physical degrees of freedom of the gravitational field, while the specific functional form of the *invariant pseudo-coordinates* matches these latter into the manifold's points. The introduction of matter has the effect of modifying the Riemann and

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[5]Actually, David Hilbert was the first person to discuss the Cauchy problem for the Einstein equations and to realize its connection to the Hole phenomenology [25]. He discussed the issue in the context of a general-relativistic generalization of Mie's special relativistic non linear electrodynamics and pointed out the necessity of fixing a special geometrically adapted (*Gaussian* in his terms, or geodesic normal as known today) coordinate system to assure the causality of the theory. In this connection see also Ref. [26]. However, as again noted by Stachel [27], Hilbert's analysis was incomplete and neglected important related problems.

[6]Actually, the first suggestion of specifying space-time points *absolutely* in terms of curvature invariants is due to Synge [31]b

Weyl tensors, namely the curvature of the 4-dimensional substratum, and to allow *measuring* the gravitational field in a geometric way for instance through effects like the geodesic deviation equation. It is important to emphasize, however, that the addition of *matter* does not modify the construction leading to the individuation of point-events, rather it makes it *conceptually more appealing*.

Finally, the procedure of individuation that we have outlined transfers, as it were, the noncommutative Poisson-Dirac structure of the Dirac observables onto the individuated point-events. The physical implications of this circumstance might deserve some attention in view of the quantization of general relativity. Some hints for the quantum gravity programme will be offered in the final Section of the paper (Concluding Survey).

A Section of the paper is devoted to our second main topic: the clarification of the concept of *Bergmann's observable* (BO) [2]. Bergmann's definition has various facets, namely a *configurational* side having to do with invariance under *passive* diffeomorphisms, an Hamiltonian side having to do with Dirac's concept of observable, and the property of *predictability* which is entangled with both sides. According to Bergmann, (his) *observables* are passive diffeomorphisms invariant quantities (PDIQ) "which can be predicted uniquely from initial data", or "quantities that are invariant under a coordinate transformation that leaves the initial data unchanged". Bergmann says in addition that they are further required to be gauge invariant, a statement that can only be interpreted as implying that Bergmann's observables are simultaneously DO. Yet, he offers no explicit demonstration of the compatibility of this bundle of statements. The clarification of this entanglement leads us to the proposal of a *main conjecture* asserting the i) existence of special Dirac's observables which are also Bergmann's observables, as well as to the ii) existence of gauge variables that are coordinate independent (namely they behave like the tetradic scalar fields of the Newman-Penrose formalism). A by-product of this achievement is the falsification of a recently advanced argument [3] asserting the absence of (any kind of) change in the observable quantities of general relativity.

Two fundamental independent but conceptually overlapping interpretational issues of general relativity have been debated in the recent literature, namely the question of the *objectivity of temporal change*, and that of *change* in general. It is well-known that many authors claim that general relativity entails the absence of objective *temporal change*. Although such claims are not *strictly* related to the issue of point individuation, if sound *in general* they could seriously weaken the conceptual appeal of our very program. Therefore we are obliged to take issue against some conclusions of this sort. As to the first point, for the sake of argument, we will restrict our remarks to the objections raised by Belot and Earmann [32] and Earman [3] (see also Refs. [33–35] as regards the so called *problem of time* or *frozen time*). According to these authors, the reduced phase space of general relativity is indeed a frozen space without evolution. We shall argue that their claim cannot have a general ontological force, essentially because is model dependent and it does not apply to all families of Einstein's space-times. We show in particular that globally-hyperbolic non-compact space-times exist - defined by suitable boundary conditions and asymptoti-

cally flat at spatial infinity [7] - that provide an explicit *counterexample* to the *frozen time argument*. The role of the generator of real time evolution in such space-times is played by the *weak* ADM energy, while the super-hamiltonian constraint has nothing to do with temporal change and is only the generator of gauge transformations connecting different admissible 3+1 splittings of space-time. We argue, therefore, that in these space-times there is neither a frozen reduced phase space nor a possible Wheeler-De Witt interpretation based on some local concept of time as in compact space-times. In conclusion, we claim that our gauge-invariant approach to general relativity is *perfectly adequate to accommodate real temporal change*, so that all the consequent developments based on it are immune to ontological criticisms like those quoted above.

There is, however, a stronger thesis about change that has been recently defended by John Earman in Ref. [3] and must be addressed separately. We shall call this thesis the *universal no-change argument*. Indeed, Earman claims that the deep structure of general relativity linked to Leibniz equivalence *and* Bergmann's definition of observable are such that:

> "Bergmann's proposal implies that there is no physical change, i.e., no change in the observable quantities, at least not for those quantities that are constructible in the most straightforward way from the materials at hand."

Note that Earman's argument is allegedly independent of Hamiltonian techniques so that it too does not menace our first topic *directly*. However, if technically viable, it would imply an inner contradiction of Bergmann's multiple definition of obervables [2] and, in particular, a flaw in the relation between Bergmann's and Dirac's notions, whose explanation is the second of our main goals.

Earman's argument is entirely based on the configurational side of the definition and depends crucially upon the property of *predictability*. The argument exploits the implications of the initial value problem of general relativity by referring to a Cauchy surface Σ_o in M^4 as specified by an intuitive geometrical representation within the Lagrangian approach. Earman's radical conclusion is that the conjunction of general covariance and *predictability* in the above Cauchy sense implies that, for any observable Bergmann's field $B(p)$, and any *active* diffeomorphism $p^{'} = \phi \cdot p$ that leaves Σ_o and its past fixed, it follows $B(p) = B(\phi^{-1} \cdot p) = B'(p) \equiv \phi^* B(p)$. Then, since Σ_o is arbitrary, the Bergmann's observable field B(p) must be *constant everywhere* in M^4.

We shall argue that this conclusion cannot be reconciled with the Hamiltonian side of Bergmann's definition of observable in that concerns *predictability*, while the Lagrangian configurational concept of *predictability* is substantially ambiguous in the case of Einstein's equations. The flaw in the argument should indeed be traced to the naive way in which the Cauchy problem is usually dealt with within the configurational Lagrangian approach.

A third result - obtained again thanks to the virtues of the Hamiltonian approach - is something new concerning the overall role of gravitational and gauge degrees of freedom.

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[7] Precisely the Christodoulou-Klainermann space-times [36] we use in this paper.

Indeed, the distinction between gauge variables and DO provided by the Shanmugadhasan [37] transformation (see Section III), conjoined with the circumstance that the Hamiltonian point of view brings naturally to a re-reading of geometrical features in terms of the traditional concept of *force*, leads to a by-product of our investigation that, again, would be extremely difficult to characterize within the *Lagrangian viewpoint* at the level of the Hilbert action or Einstein's equations. The additional by-product is something that should be added to the traditional wisdom of the equivalence principle asserting the local impossibility of distinguishing gravitational from inertial effects. Actually, the isolation of the gauge arbitrariness from the true intrinsic degrees of freedom of the gravitational field is instrumental to understand and visualize which aspects of the local effects, showing themselves on test matter, have a *genuine gravitational origin* and which aspects depend solely upon the choice of the (local) reference frame and could therefore even be named *inertial* in analogy with their non-relativistic Newtonian counterparts. Indeed, two main differences characterize the issue of *inertial effects* in general relativity with respect to the non-relativistic situation: the existence of *autonomous degrees of freedom* of the gravitational field independently of the presence of matter sources, on the one hand, and the *local nature of the general-relativistic reference systems*, on the other. We shall show that, although the very definition of *inertial forces* (and of *gravitational force* in general) is rather arbitrary in general relativity, it appears natural to characterize first of all as genuine gravitational effects those which are directly correlated to the DO, while the gauge variables appear to be correlated to the general relativistic counterparts of Newtonian inertial effects.

Another aspect of the Hamiltonian connection "*gauge variables - inertial effects*" is related to the 3+1 splitting of space-time required for the canonical formalism. Each splitting is associated with a foliation of space-time whose leaves are *Cauchy simultaneity* space-like hyper-surfaces. While the field of unit normals to these surfaces identifies a *surface-forming congruence of time-like observers*, the field of the evolution vectors identifies a *rotating congruence of time-like observers*. Since a variation of the gauge variables modifies the foliation, the identification of the two congruences of time-like observers is connected to the fixation of the gauge, namely, on-shell, to the choice of 4-coordinates. Then a variation of gauge variables also modifies the inertial effects.

It is clear by now that a complete gauge fixing within canonical gravity has the following implications: i) the choice of a unique 3+1 splitting with its associated foliation; ii) the choice of well-defined congruences of time-like observers; iii) the *on-shell* choice of a unique 4-coordinate system. In physical terms this set of choices amount to choosing a *network of intertwined and synchronized local laboratories made up with test matter* (obviously up to a coherent choice of chrono-geometric standards). This interpretation shows that, unlike in ordinary gauge theories where the gauge variables are inessential degrees of freedom, the concept of reduced phase space is very abstract and not directly useful in general relativity: it is nothing else than the space of gravitational equivalence classes each of which is described by the set of all laboratory networks living in a gauge orbit. This makes the requirement of an *intrinsic* characterization for the reduced phase space asked by Belot and Earman [32] rather meaningless.

The only weakness of the previous distinction is that the separation of the two autonomous degrees of freedom of the gravitational field from the gauge variables is, as yet, a coordinate (i.e. gauge) - dependent concept. The known examples of pairs of conju-

gate DO are neither coordinate-independent (they are not PDIQ) nor tensors. Bergmann asserts that the only known method (at the time) to build BO is based on the existence of Bergmann-Komar invariant pseudo-coordinates. The results of this method, however, are of difficult interpretation, so that, in spite of the importance of this alternative non-Hamiltonian definition of observables, no explicit determination of them has been proposed so far. A possible starting point to attack the problem of the connection of DO with BO seems to be a Hamiltonian reformulation of the Newman-Penrose formalism [38] (it contains only PDIQ) employing Hamiltonian null-tetrads carried by the time-like observers of the congruence orthogonal to the admissible space-like hyper-surfaces. This suggests the technical *conjecture* that special Darboux bases for canonical gravity should exist in which the inertial effects (gauge variables) are described by PDIQ while the autonomous degrees of freedom (DO) are *also* BO. Note that, since Newman-Penrose PDIQ are tetradic quantities, the validity of the conjecture would also eliminate the existing difference between the observables for the gravitational field and the observables for matter, built usually by means of the tetrads associated to some time-like observer. Furthermore, this would also provide a starting point for defining a metrology in general relativity in a generally covariant way[8], replacing the empirical metrology [39] used till now. It would also enable to replace by dynamical matter the *test matter* of the axiomatic approach to measurement theory (see Appendix C).

A final step of our analysis consists in suggesting how the physical individuation of space-time points, introduced at the conceptual level, could in principle be implemented with *a well-defined empirical procedure, an experimental set-up and protocol for positioning and orientation*. This suggestion is outlined in correspondence with the abstract treatment of the empirical foundation of general relativity as exposed in the classical paper of Ehlers, Pirani and Schild [40]. The conjunction of the Hamiltonian treatment of the initial value problem, with the correlated physical individuation of space-time points, and the practice of general-relativistic measurement, on the backdrop of the axiomatic foundation closes, as it were, the *coordinative circuit* of general relativity.

The plan of the paper is the following. In Section II the Einstein-Hilbert Lagrangian viewpoint and the related local symmetries are summarized. Particular emphasis is given to the analysis of the most general group Q of dynamical symmetries of Enstein's equations (Bergmann-Komar group), and the *passive view* of *active* diffeomorphisms is clarified. The ADM Hamiltonian viewpoint and its related canonical local symmetries are expounded in Section III. Building on the acquired knowledge about the structure of Q, particular emphasis is given to a discussion of the general Hamiltonian gauge group and to the correspondence between *active diffeomorphisms* and *on-shell gauge transformations*. Section IV is devoted to the central issue of the individuation of the *mathematical points* of M^4 as *physical point-events* by means of a peculiar gauge-fixing to Bergmann-Komar *intrinsic pseudo-coordinates*. A digression on the concept of BO and the criticism of the *frozen time* and *universal no-change* arguments are the content of Section V where our *main conjecture* is advanced

[8]Recall that this is the main conceptual difference from the non-dynamical metrology of special relativity

concerning the relations between DO and BO. The results obtained in Sections III, IV and V about the canonical reduction lead naturally to the physical interpretation of the DO and the gauge variables as characterizing *tidal-like* and *inertial-like* effects, respectively: this is discussed in Section VI. An outline of the empirical *coordinative circuit* of general relativity is sketched in Section VII. A Concluding Survey containing some hints in view of the quantum gravity programme and three Appendices complete the paper.

II. THE EINSTEIN-HILBERT LAGRANGIAN VIEWPOINT AND THE RELATED LOCAL SYMMETRIES.

Given a pseudo-Riemannian 4-dimensional manifold M^4, the Einstein-Hilbert action for pure gravity without matter [9]

$$S_H = \int d^4x\, \mathcal{L}(x) = \int d^4x\, \sqrt{{}^4g}\, {}^4R, \tag{2.1}$$

defines a variational principle for the metric 2-tensor ${}^4g_{\mu\nu}(x)$ over M^4. The associated Euler-Lagrange equations are Einstein's equations

$${}^4G_{\mu\nu}(x) \stackrel{def}{=} {}^4R_{\mu\nu}(x) - \frac{1}{2}\, {}^4R(x)\, {}^4g_{\mu\nu}(x) = 0. \tag{2.2}$$

Although - as clarified in the Introduction - we should look at the above equations as pure *mathematical* relations, we will discuss the counting of the degrees of freedom following the procedure applied in any classical singular field theory.

As well known, the action (2.1) is invariant under general coordinate transformations. In other words, the *passive diffeomorphisms* $_PDiff\, M^4$ are *local Noether symmetries* (second Noether theorem) of the action. This has the consequence that:

i) Einstein's equations are form invariant under coordinate transformations (a property usually named *general covariance*);

ii) the Lagrangian density $\mathcal{L}(x)$ is singular, namely its Hessian matrix has vanishing determinant.

This in turn entails that:

i) four of the ten Einstein equations are *Lagrangian constraints*, namely restrictions on the Cauchy data;

ii) four combinations of Einstein's equations and their gradients vanish identically (*Bianchi identities*).

In conclusion, there are only two *dynamical* second-order equations depending on the *accelerations* of the metric tensor. As a consequence, the ten components ${}^4g_{\mu\nu}(x)$ of the metric tensor are functionals of two "*deterministic*" dynamical degrees of freedom and eight further degrees of freedom which are left completely undetermined by Einstein's equations *even once the Lagrangian constraints are satisfied*. This state of affairs makes the treatment of both the Cauchy problem of the non-hyperbolic system of Einstein's equations and the definition of observables extremely complicated within the Lagrangian context [28]. Let us stress that precisely this arbitrariness is the source of what has been interpreted as *indeterminism* of general relativity in the debate about the Hole Argument.

Since passive diffeomorphisms play the role of Lagrangian gauge transformations, *a complete Lagrangian gauge fixing amounts to a definite choice of the coordinates on* M^4, a choice

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[9] x^μ is a coordinate chart in the atlas of M^4.

which, on the other hand, is necessary in order to explicitly solve the Einstein partial differential equations.

On the other hand, in modern terminology, general covariance implies that *a physical solution of Einstein's equations* properly corresponds to a *4-geometry*, namely the equivalence class of all the 4-metric tensors, solutions of the equations, written in all possible 4-coordinate systems. This equivalence class is usually represented by the quotient ${}^4Geom = {}^4Riem/{}_PDiff\, M^4$, where ${}^4Riem$ denotes the space of metric tensors solutions of Einstein's equations. Then, any two *inequivalent* Einstein space-times are different 4-geometries.

Besides local Noether symmetries of the action, Einstein's equations, considered as a set of partial differential equations, have their own *dynamical symmetries* [41] which only partially overlap with the former ones. Let us stress that:

i) a dynamical symmetry is defined only on the space of solutions of the equations of motion, namely it is an *on-shell* concept;

ii) only a subset of such symmetries (called *Noether dynamical symmetries*) can be extended *off-shell* in the variational treatment of action principles. The *passive diffeomorphisms* ${}_PDiff\, M^4$ are just an instantiation of Noether dynamical symmetries of Einstein's equations.

iii) among the dynamical symmetries of Einstein's equations, there are all the *active diffeomorphisms* ${}_ADiff\, M^4$, which, as said in the Introduction, are the essential core of Einstein's *Hole Argument* [10].

Yet, according to Stachel [8], it is just the dynamical symmetry nature of *active diffeomorphisms* that expresses the real *physically relevant* content of *general covariance*. On the other hand the natural Noether symmetries of the Hilbert action are the *passive diffeomorphisms*. This dualism active-passive has been a continuous source of confusion and ambiguity in the literature. We claim, however, that a clarification of the issue can be drawn from a nearly forgotten paper by Bergmann and Komar [11] [11] in which it is shown that the biggest group Q of *passive dynamical symmetries* of Einstein's equations is not ${}_PDiff\, M^4$ $[x^{'\,\mu} = f^\mu(x^\nu)]$ but instead a larger group of transformations of the form

$$Q : x^{'\,\mu} = f^\mu(x^\nu, {}^4g_{\alpha\beta}(x)), \tag{2.3}$$

which, of course is thereby extended to tensors in the standard way: ${}^4g^{'}_{\mu\nu}(x^{'}) = \frac{\partial x^\alpha}{\partial f^\mu} \frac{\partial x^\beta}{\partial f^\nu}\, {}^4g_{\alpha\beta}(x)$. It is clear that in this way we allow for metric dependent coordinate systems, whose associated 4-metrics are in general different from those obtainable from a given

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[10]Note that a subset of *active diffeomorphisms* are the *conformal isometries*, i.e. those conformal transformations which are also active diffeomorphisms, namely ${}^4\tilde g = \Omega^2\, {}^4g \equiv \phi^*\, {}^4g$ for some $\phi \in {}_ADiff\, M^4$ with Ω strictly positive. Since the Hilbert action is not invariant under the conformal transformations which are not *ordinary isometries* (i.e. conformal isometries with $\Omega = 1$ for which $\mathcal{L}_X\, {}^4g = 0$, if X is the associated Killing vector field), only these latter are Noether dynamical symmetries.

[11]See, however, Ref. [42].

4-metric solution of Einstein's equations by *passive diffeomorphisms*: actually, these transformations map points to points, but associate with a given point x an image point x' that depends also on the metric field [12]. It is remarkable, however, that not only these new transformed 4-metric tensors are still solutions of Einstein's equations, but that indeed they belong to the it same 4-geometry, i.e. the same equivalence class generated by applying all it passive diffeomorphisms to the original 4-metrics: ${}^4Geom = {}^4Riem/Q = {}^4Riem/{}_PDiff\, M^4$. Note, incidentally, that this circumstance is mathematically possible only because ${}_PDiff\, M^4$ is a *non-normal* sub-group of Q. The 4-metrics built by using passive diffeomorphisms are, as it were, a dense sub-set of the metrics obtainable by means of the group Q.

There is no clear statement in the literature about the dynamical symmetry status of the group ${}_ADiff\, M^4$ of *active diffeomorphisms* and their relationship with the group Q, a point which is fundamental for our program. To clarify this point, let us consider an infinitesimal transformation of the type (2.3) connecting a 4-coordinate system $[x^\mu]$ to a new one $[x^{'\mu}]$ by means of metric-dependent infinitesimal descriptors:

$$x^{'\,\mu} = x^\mu + \delta\, x^\mu = x^\mu + \xi^\mu(x, {}^4g). \tag{2.4}$$

This will induce the usual formal variation of the metric tensor [13]

$$\bar{\delta}\, {}^4g_{\mu\nu} = -\Big(\xi_{\mu;\nu}(x, {}^4g) + \xi_{\nu;\mu}(x, {}^4g)\Big). \tag{2.5}$$

If $\bar{\delta}\, {}^4g_{\mu\nu}(x)$ is now identified with the local variation of the metric tensor induced by the *drag along* of the metric under an infinitesimal active diffeomorphism ${}^4g \mapsto {}^4\tilde{g}$ so that

$$\bar{\delta}\, {}^4g_{\mu\nu} =\equiv {}^4\tilde{g}_{\mu\nu}(x) - {}^4g_{\mu\nu}(x) = -\Big(\xi_{\mu;\nu}(x, {}^4g) + \xi_{\nu;\mu}(x, {}^4g)\Big), \tag{2.6}$$

the solution $\xi_\mu(x, {}^4g)$ of these Killing-type equations identifies a corresponding *passive* Bergmann-Komar dynamical symmetry belonging to Q. We see that, just as said in the Introduction, the new system of coordinates $[x^{'\mu}]$ is identified with the *drag along* of the old coordinate system.

This result should imply that *all the active diffeomorphisms connected with the identity in ${}_ADiff\, M^4$ can be reinterpreted as elements of a non-normal sub-group of generalized passive transformations in Q*. Clearly this sub-group is disjoint from the sub-group ${}_PDiff\, M^4$: this in turn is possible because diffeomorphism groups do not possess a canonical identity. Let us recall, however, that unfortunately there is no viable mathematical treatment of the diffeomorphism group in the large.

What has been defined in the Introduction as (Earman-Norton) Leibniz equivalence of metric tensors [4] means that an *Einstein (or on-shell, or dynamical) gravitational field* is an equivalence class of solutions of Einstein's equation *modulo* the dynamical symmetry transformations of ${}_ADiff\, M^4$. Therefore, we also have

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[12]Strictly speaking, Eqs.(2.3) should be defined as transformations on the tensor bundle over M^4.

[13]What is relevant here is the *local* variation $\bar{\delta}\, {}^4g_{\mu\nu}(x) = \mathcal{L}_{-\xi^\gamma\,\partial_\gamma}\, {}^4g_{\mu\nu}(x) = {}^4g^{'}_{\mu\nu}(x) - {}^4g_{\mu\nu}(x)$ which differs from the *total* variation by a *convective* term: $\delta\, {}^4g_{\mu\nu}(x) = {}^4g^{'}_{\mu\nu}(x^{'}) - {}^4g_{\mu\nu}(x) = \bar{\delta}\, {}^4g_{\mu\nu}(x) + \delta\, x^\gamma\, \partial_\gamma\, {}^4g_{\mu\nu}(x)$.

$$^4Geom = {}^4Riem/_A Diff\, M^4 = {}^4Riem/Q = {}^4Riem/_P Diff\, M^4. \tag{2.7}$$

It is clear that a parametrization of the 4-geometries should be grounded on the two independent dynamical degrees of freedom of the gravitational field. Within the framework of the Lagrangian dynamics, however, no algorithm is known for evaluating the observables of the gravitational field, viz. its two independent degrees of freedom. The only result we know of is given in Ref. [36] where, after a study of the index of Einstein's equations, it is stated that the two degrees of freedom are locally associated to *symmetric trace-free 2-tensors on two-planes*, suggesting a connection with the Newman-Penrose formalism [38].

On the other hand, as we shall see in the next Section, it is the Hamiltonian framework which has the proper tools to attack these problems. Essentially, this is due to the fact that the Hamiltonian methods allow to work *off-shell*, i.e., without immediate transition to the space of solutions of Einstein's equations. Thus the soldering to the above results is reached *only* at the end of the canonical reduction, when the *on-shell* restriction is made [14].

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[14]Note nevertheless that even at the Lagrangian level one can define *off-shell (or kinematical) gravitational fields* defined as ${}^4Riem'/_P Diff\, M^4$, where ${}^4Riem'$ are all the possible metric tensors on M^4. Of course only the subset of solutions of Einstein equations are Einstein gravitational fields.

III. THE ADM HAMILTONIAN VIEWPOINT AND THE RELATED CANONICAL LOCAL SYMMETRIES.

This Section provides the technical analysis of the Cauchy problem and the counting of degrees of freedom within the framework of the ADM canonical formulation of metric gravity [43]. Recall that this formulation holds for globally hyperbolic pseudo-Riemannian 4-manifolds M^4 which are asynptotically flat at spatial infinity [15]. Unlike the Lagrangian formulation, the Hamiltonian formalism requires a 3+1 splitting of M^4 and a global time function τ. This entails in turn a foliation of M^4 by space-like hyper-surfaces Σ_τ (*simultaneity Cauchy surfaces*), to be coordinatized by *adapted* 3-coordinates $\vec{\sigma}$ [16]. A canonical formulation with well-defined Poisson brackets requires in addition the specification of suitable *boundary conditions* at spatial infinity, viz. a definite choice of the functional space for the fields[17]. Finally, one should not forget the fact that the problem of the boundary conditions constitutes an intriguing issue within the Lagrangian approach.

The reader is referred to Appendix A for the relevant notations and technical developments of the Hamiltonian description of metric gravity, which requires the use of Dirac-Bergmann [45–49] theory of constraints (see Refs. [50,51] for updated reviews).

We start off with replacement of the ten components ${}^4g_{\mu\nu}$ of the 4-metric tensor by the

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[15] As shown in Ref. [44] and said in the Introduction, in this case the so-called *problem of time* can be treated in such a way that in presence of matter and in the special-relativistic limit of vanishing Newton constant, one recovers the parametrized Minkowski theories equipped with a *global time*: in such theories, however, space-time points are individuated as point-events in a kinematical and absolute way. Of course, this result is precluded if space-time is *spatially compact without boundary (or closed)*. Let us remark that parametrized Minkowski theories give the reformulation of the dynamics of isolated systems in special relativity on arbitrary space-like hyper-surfaces, leaves of the foliation associated with an arbitrary 3+1 splitting and defining a surface-forming congruence of accelerated time-like observers. In these theories the embeddings $z^\mu(\tau, \sigma)$ of the space-like hyper-surfaces are new configuration variables at the Lagrangian level. However they are *gauge variables* because the Lagrangian is invariant under separate τ- and $\vec{\sigma}$-reparametrizations (which are diffeomorphisms). This form of *special relativistic general covariance* implies the existence of four first class constraints analogous to the super-hamitonian and super-momentum constraints of ADM canonical gravity, playing the role of assuring the independence of the description from the choice of the 3+1 splitting (there is no Hole phenomenology)

[16] An improper vector notation is used throughout for the sake of simplicity.

[17] See Ref. [44] for a detailed discussion of this point. It is pointed out there that in order to have well defined asymptotic *weak and strong ADM Poincare' charges* (generators of the asymptotic Poincare' group) all fields must have a suitable direction-independent limit at spatial infinity. In presence of matter, switching off the Newton constant reduces these charges to the conserved generators of the Poincare' group for the isolated system with the same matter. Of course, in closed space-times, the ADM Poincare' charges do not exist and the special relativistic limit is lost.

configuration variables of ADM canonical gravity: the *lapse* $N(\tau,\vec{\sigma})$ and *shift* $N_r(\tau,\vec{\sigma})$ functions and the six components of the *3-metric tensor* on Σ_τ, ${}^3g_{rs}(\tau,\vec{\sigma})$. Einstein's equations are then recovered as the Euler-Lagrange equations of the ADM action

$$\begin{aligned} S_{ADM} &= \int d\tau\, L_{ADM}(\tau) = \int d\tau d^3\sigma \mathcal{L}_{ADM}(\tau,\vec{\sigma}) = \\ &= -\epsilon k \int_{\triangle\tau} d\tau \int d^3\sigma\, \{\sqrt{\gamma} N\, [{}^3R + {}^3K_{rs}\, {}^3K^{rs} - ({}^3K)^2]\}(\tau,\vec{\sigma}), \end{aligned} \tag{3.1}$$

which differs from Einstein-Hilbert action (2.1) by a suitable surface term. Here ${}^3K_{rs}$ is the extrinsic curvature of Σ_τ, ${}^3K$ its trace, and ${}^3R$ the 3-curvature scalar.

Besides the ten configuration variables listed above, the ADM functional phase space Γ_{20} is *coordinatized* by ten canonical momenta $\tilde{\pi}^N(\tau,\vec{\sigma})$, $\tilde{\pi}^r_{\vec{N}}(\tau,\vec{\sigma})$, ${}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma})$ [18]. Such canonical variables, however, are not independent since they are restricted to the *constraint sub-manifold* Γ_{12} by the eight *first class* constraints

$$\begin{aligned} &\tilde{\pi}^N(\tau,\vec{\sigma}) \approx 0, \\ &\tilde{\pi}^r_{\vec{N}}(\tau,\vec{\sigma}) \approx 0, \\ &\tilde{\mathcal{H}}(\tau,\vec{\sigma}) = \epsilon[k\sqrt{\gamma}\, {}^3R - \frac{1}{2k\sqrt{\gamma}} {}^3G_{rsuv}\, {}^3\tilde{\Pi}^{rs}\, {}^3\tilde{\Pi}^{uv}](\tau,\vec{\sigma}) \approx 0, \\ &{}^3\tilde{\mathcal{H}}^r(\tau,\vec{\sigma}) = -2\, {}^3\tilde{\Pi}^{rs}{}_{|s}(\tau,\vec{\sigma}) = -2[\partial_s\, {}^3\tilde{\Pi}^{rs} + {}^3\Gamma^r_{su}\, {}^3\tilde{\Pi}^{su}](\tau,\vec{\sigma}) \approx 0. \end{aligned} \tag{3.2}$$

While the first four are *primary* constraints, the remaining four are the super-hamiltonian and super-momentum *secondary* constraints arising from the requirement that the primary constraints be constant in τ. More precisely, this requirement guarantees that, once we have chosen the initial data inside the constraint sub-manifold $\Gamma_{12}(\tau_o)$ corresponding to a given initial Cauchy surface Σ_{τ_o}, the time evolution does not take them out of the constraint sub-manifolds $\Gamma_{12}(\tau)$, for $\tau > \tau_o$.

The evolution in τ is ruled by the Hamilton-Dirac Hamiltonian

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[18] As shown in Ref. [44], a consistent treatment of the boundary conditions at spatial infinity requires the explicit separation of the *asymptotic* part of the lapse and shift functions from their *bulk* part: $N(\tau,\vec{\sigma}) = N_{(as)}(\tau,\vec{\sigma}) + n(\tau,\vec{\sigma})$, $N_r(\tau,\vec{\sigma}) = N_{(as)r}(\tau,\vec{\sigma}) + n_r(\tau,\vec{\sigma})$, with n and n_r tending to zero at spatial infinity in a direction-independent way. On the contrary, $N_{(as)}(\tau,\vec{\sigma}) = -\lambda_\tau(\tau) - \frac{1}{2}\lambda_{\tau u}(\tau)\,\sigma^u$ and $N_{(as)r}(\tau,\vec{\sigma}) = -\lambda_r(\tau) - \frac{1}{2}\lambda_{ru}(\tau)\,\sigma^u$. The *Christodoulou-Klainermann space-times* [36], with their *rest-frame* condition of zero ADM 3-momentum and absence of super-translations, are singled out by these considerations. The allowed foliations of these space-times tend asymptotically to Minkowski hyper-planes in a direction-independent way and are asymptotically orthogonal to the ADM four-momentum. They have $N_{(as)}(\tau,\vec{\sigma}) = \epsilon$, $N_{(as)r}(\tau,\vec{\sigma}) = 0$. Therefore, in these space-times there are *asymptotic inertial time-like observers* (the *fixed stars* or the *CMB rest frame*) and the global mathematical time labeling the Cauchy surfaces can be identified with their rest time. For the sake of simplicity we shall ignore these aspects of the theory, with the caveat that the canonical pairs N, $\tilde{\pi}^N$, N_r, $\tilde{\pi}^r_{\vec{N}}$ should be always replaced by the pairs n, $\tilde{\pi}^n$, n_r, $\tilde{\pi}^r_{\vec{n}}$.

$$H_{(D)ADM} = \int d^3\sigma \Big[N\,\tilde{\mathcal{H}} + N_r\,{}^3\tilde{\mathcal{H}}^r + \lambda_N\,\tilde{\pi}^N + \lambda^{\vec{N}}_r\,\tilde{\pi}^r_{\vec{N}} \Big](\tau,\vec{\sigma}) \approx 0, \tag{3.3}$$

where $\lambda_N(\tau,\vec{\sigma})$ and $\lambda^r_{\vec{N}}(\tau,\vec{\sigma})$ are *arbitrary Dirac multipliers* in front of the primary constraints[19]. The resulting hyperbolic system of Hamilton-Dirac equations has the same solutions of the non-hyperbolic system of (Lagrangian) Einstein's equations with the same boundary conditions. Let us stress that the Hamiltonian hyperbolicity is explicitly paid by the arbitrariness of the Dirac multipliers. Of course this is just the Hamiltonian counterpart of the "*indeterminateness*" or the so-called "*indeterminism*" surfacing in the Hole Argument.

At this point a number of important questions must be clarified. When used as generators of canonical transformations, the eight first class constraints will map points of the constraint surface to points on the same surface. We shall say that they generate the infinitesimal transformations of the *off-shell Hamiltonian gauge group* $\mathcal{G}_8$ [20]. The action of $\mathcal{G}_8$ gives rise to a *Hamiltonian gauge orbit* through each point of the constraint sub-manifold Γ_{12}. Every such orbit is parametrized by eight phase space functions - namely the independent *off-shell Hamiltonian gauge variables* - conjugated to the first class constraints. We are left thereby with a pair of conjugate canonical variables, the *off-shell DO*, which are the only *Hamiltonian gauge-invariant and "determinatedly" ruled* quantities. The same counting of degrees of freedom of the Lagrangian approach is thus obtained. Finally, let us stress here, in view of the later discussion, that both the off-shell Christoffel symbols and the off-shell Riemann tensor can be read as functions of both the off-shell DO and the Hamiltonian gauge variables. Likewise, the on-shell Christoffel symbols and the on-shell Riemann tensor will depend on both the on-shell DO and the on-shell Hamiltonian gauge variables.

The eight infinitesimal off-shell Hamiltonian gauge transformations have the following interpretation [44]:

i) those generated by the four primary constraints modify the lapse and shift functions: these in turn determine how densely the space-like hyper-surfaces Σ_τ are distributed in space-time and also the conventions to be made on each Σ_τ about simultaneity (the choice of clocks synchronization) and gravito-magnetism;

ii) those generated by the three super-momentum constraints induce a transition on Σ_τ from a given 3-coordinate system to another one;

iii) that generated by the super-hamiltonian constraint induces a transition from a given 3+1 splitting of M^4 to another one, by operating normal deformations [52] of the space-like hyper-surfaces[21].

---

[19] These are four *velocity functions* (gradients of the metric tensor) which are not determined by Einstein's equations. As shown in Ref. [44], the correct treatment of the boundary conditions leads to rewrite Eq.(3.3) in terms of n and n_r (see footnotes 17 and 18).

[20] Note that the off-shell Hamiltonian gauge transformations are *local Noether transformations* (second Noether theorem) under which the ADM Lagrangian (3.1) is *quasi-invariant*.

[21] Note that in *compact* space-times the super-hamiltonian constraint is usually interpreted as generator of the evolution in some *internal time*, either like York's internal *extrinsic* time or like

Making the quotient of the constraint hyper-surface with respect to the off-shell Hamiltonian gauge transformations by defining $\Gamma_4 = \Gamma_{12}/\mathcal{G}_8$, we obtain the so-called *reduced off-shell conformal super-space*. Each of its points, i.e. a *Hamiltonian off-shell (or kinematical) gravitational field*, is an off-shell equivalence class, called an *off-shell conformal 3-geometry*, for the space-like hyper-surfaces Σ_τ: note that, since it contains all the off-shell 4-geometries connected by Hamiltonian gauge transformations, *it is not a 4-geometry.*

An important digression is in order here. The space of parameters of the off-shell gauge group $\mathcal{G}_8$ contains eight arbitrary functions. Four of them are the Dirac multipliers $\lambda_N(\tau, \vec{\sigma})$, $\lambda_r^{\vec{N}}(\tau, \vec{\sigma})$ of Eqs.(3.3), while the other four are functions $\alpha(\tau, \vec{\sigma})$, $\alpha_r(\tau, \vec{\sigma})$ which generalize the lapse and shift functions in front of the secondary constraints in Eqs.(3.3) [22]. These arbitrary functions correspond to the eight local Noether symmetries under which the ADM action is quasi-invariant.

On the other hand, from the analysis of the dynamical symmetries of the Hamilton equations (equivalent to Einstein's equations), it turns out (see Refs. [53,54]) that *on-shell* only a sub-group $\mathcal{G}_{4\,dyn}$ of $\mathcal{G}_8$ survives, depending on four arbitrary functions only. But in the present context, a crucial result for our subsequent discussion is that a further subset, that we will denote by $\mathcal{G}_{4\,P} \subset \mathcal{G}_{4\,dyn}$, can be identified within the sub-group $\mathcal{G}_{4\,dyn}$: precisely the subset corresponding to the phase space counterparts of those passive diffeomorphisms which are *projectable* to phase space. On the other hand, as already said, Einstein's equations have Q as the largest group of dynamical symmetries and, even if irrelevant to the local Noether symmetries of the ADM action, this larger group is of fundamental importance for our considerations. In order to take it into account in the present context, the parameter space of $\mathcal{G}_8$ must be enlarged to arbitrary functions depending also on the 3-metric, $\lambda_N(\tau, \vec{\sigma}) \mapsto \lambda_N(\tau, \vec{\sigma}, {}^3g_{rs}(\tau, \vec{\sigma}))$, ... , $\alpha_r(\tau, \vec{\sigma}) \mapsto \alpha_r(\tau, \vec{\sigma}, {}^3g_{rs}(\tau, \vec{\sigma}))$. Then, the restriction of this enlarged gauge group to the *dynamical symmetries* of the Hamilton equations defines an extended group $\tilde{\mathcal{G}}_{4\,dyn}$ which, under inverse Legendre transformation, defines a new *non-normal* sub-group Q_{can} of the group Q (see Ref. [14]). But now, the remarkable and fundamental point is that Q_{can} *contains both active and passive diffeomorphisms*. In particular:

i) the intersection $Q_{can} \cap {}_PDiff\, M^4$ identifies the space-time passive diffeomorphisms which, respecting the 3+1 splitting of space-time, are *projectable* to $\mathcal{G}_{4\,P}$ in phase space;

ii) the remaining elements of Q_{can} are the *projectable* subset of active diffeomorphisms in their passive view.

This entails that, as said in Ref. [14], Eq.(2.4) may be completed with ${}^4Geom = {}^4Riem/Q_{can}$.

Misner's internal *intrinsic* time. Here instead the super-hamiltonian constraint is the generator of those Hamiltonian gauge transformations which imply that the description is independent of the choice of the allowed 3+1 splitting of space-time: *this is the correct answer to the criticisms raised against the phase space approach on the basis of its lack of manifest covariance.*

[22] In Ref. [14] they are called *descriptors* and written in the form $\alpha = N\,\xi$, $\alpha^r = {}^3g^{rs}\,\alpha_s = \xi^r \pm N^r\,\xi$.

In conclusion, the real gauge group acting on the space of the solutions of the Hamilton-Dirac equations (A16) is the *on-shell extended Hamiltonian gauge group* $\tilde{\mathcal{G}}_{4\,dyn}$ and the on-shell equivalence classes obtained by making the quotient with respect to it eventually coincide with the on-shell 4-geometries of the Lagrangian theory. Therefore, the *Hamiltonian Einstein (or on-shell, or dynamical) gravitational fields* coincide with the Lagrangian Einstein (or on-shell, or dynamical) gravitational fields.

This is the way in which *passive* space-time diffeomorphisms, under which the Hilbert action is invariant, are reconciled *on-shell* with the allowed Hamiltonian gauge transformations adapted to the 3+1 splittings of the ADM formalism. Furthermore, our analysis of the Hamiltonian gauge transformations and their Legendre counterparts gives an extra *bonus*: namely that the on-shell phase space extended gauge transformations include also symmetries that are images of *active* space-time diffeomorphsms. The basic relevance of this result for a deep understanding of the Hole Argument will appear fully in Section IV.

Having clarified these important issues, let us come back to the canonical reduction. The off-shell freedom corresponding to the eight independent types of Hamiltonian gauge transformations is reduced on-shell to four types like in the case of ${}_PDiff\; M^4$: precisely the transformations in $[Q_{can} \cap {}_PDiff\; M^4]$. At the off-shell level, this property is manifest by the circumstance that the original Dirac Hamiltonian contains *only* 4 arbitrary Dirac multipliers and that the *correct gauge-fixing procedure* [55,44] starts by giving *only* the four gauge fixing constraints for the secondary constraints. The requirement of time constancy then generates the four gauge fixing constraints to the primary constraints, while time constancy of such secondary gauge fixings leads to the determination of the four Dirac multipliers[23]. Since the original constraints plus the above eight gauge fixing constraints form a second class set, it is possible to introduce the associated *Dirac brackets* and conclude the canonical reduction by realizing an off-shell reduced phase space Γ_4. Of course, once we reach a *completely fixed Hamiltonian gauge* (a copy of Γ_4), general covariance is completely broken. Finally, note that a completely fixed Hamiltonian gauge *on-shell* is equivalent to a *definite choice of the space-time 4-coordinates* on M^4 within the Lagrangian viewpoint.

In order to visualize the meaning of the various types of degrees of freedom [24] we need a determination of a *Shanmugadhasan canonical basis* [37] of metric gravity [44] having the following structure ($\bar{a} = 1, 2$ are non-tensorial indices of the DO [25] $r_{\bar{a}}$, $\pi_{\bar{a}}$) with

$$
\begin{array}{|l|l|l|}
\hline
N & N_r & {}^3g_{rs} \\
\hline
\tilde{\pi}^N \approx 0 & \tilde{\pi}^r_{\vec{N}} \approx 0 & {}^3\tilde{\Pi}^{rs} \\
\hline
\end{array}
\qquad \longrightarrow \qquad
\begin{array}{|l|l|l|l|}
\hline
N \qquad\quad N_r & \xi^r & \phi & r_{\bar{a}} \\
\hline
\tilde{\pi}^N \approx 0 \;\; \tilde{\pi}^r_{\vec{N}} \approx 0 & \tilde{\pi}^{\vec{\mathcal{H}}}_r \approx 0 & \pi_\phi & \pi_{\bar{a}} \\
\hline
\end{array}
$$

---

[23]This agrees with the results of Ref. [56] according to which the *projectable* space-time diffeomorphisms depend only on four arbitrary functions and their time derivatives.

[24]This visualization remains only implicit in the conformal Lichnerowicz-York approach [57–60].

[25]Let us stress that the DO are in general neither tensors nor invariants under space-time diffeomorphisms. Therefore their (unknown) functional dependence on the original variables changes (off-shell) with the gauge and, therefore, (on-shell) with the 4-coordinate system.

$$
\longrightarrow \quad
\begin{array}{|ll|l|l|l|}
\hline
N & N_r & \xi^r & Q_{\mathcal{H}} \approx 0 & r^{'}_{\bar a} \\
\hline
\tilde \pi^N \approx 0 & \tilde \pi^r_{\vec N} \approx 0 & \tilde \pi^{\vec{\mathcal{H}}}_r \approx 0 & \Pi_{\mathcal{H}} & \pi^{'}_{\bar a} \\
\hline
\end{array}
\qquad (3.4)
$$

It is seen that we need a sequence of two canonical transformations.

a) The first one replaces seven first-class constraints with as many Abelian momenta (ξ^r are the gauge parameters of the passive 3-diffeomorphisms generated by the supermomentum constraints) and introduces the conformal factor ϕ of the 3-metric as the configuration variable to be determined by the super-hamiltonian constraint [26]. Note that the final gauge variable, namely the momentum π_ϕ conjugate to the conformal factor, is the only gauge variable of momentum type: it plays the role of a *time* variable, so that the Lorentz signature of space-time is made manifest by the Shanmugadhasan transformation in the set of gauge variables $(\pi_\phi; \xi^r)$; this makes the difference with respect to the proposals of Refs. [33,61]. More precisely, the first canonical transformation should be called a *quasi-Shanmugadhasan* transformation, because nobody has succeeded so far in Abelianizing the super-hamiltonian constraint. Note furthermore that this transformation is a *point* canonical transformation.

b) The second canonical transformation would be instead a *complete Shanmugadhasan* transformation, where $Q_{\mathcal{H}}(\tau, \vec \sigma) \approx 0$ would denote the Abelianization of the super-hamiltonian constraint [27]. The variables N, N_r, ξ^r, $\Pi_{\mathcal{H}}$ are the final *Abelianized Hamiltonian gauge variables* and $r^{'}_{\bar a}$, $\pi^{'}_{\bar a}$ the final DO.

In absence of explicit solutions of the Lichnerowicz equation, the best we can do is to

---

[26] Recall that the *strong* ADM energy is the flux through the surface at spatial infinity of a function of the 3-metric only, and it is weakly equal to the *weak* ADM energy (volume form) which contains all the dependence on the ADM momenta. This implies [44] that the super-hamiltonian constraint must be interpreted as the equation (*Lichnerowicz equation*) that uniquely determines the *conformal factor* $\phi = (det\, {}^3g)^{1/12}$ of the 3-metric as a functional of the other variables. This means that the associated gauge variable is the *canonical momentum* π_ϕ *conjugate to the conformal factor*: this latter carries information about the extrinsic curvature of Σ_τ. It is just this variable, and *not* York's time, which parametrizes the *normal* deformation of the embeddable space-like hyper-surfaces Σ_τ. As said in footnote 15, the theory is independent of the choice of the 3+1 splitting like in parametrized Minkowski theories. As a matter of fact, a gauge fixing for the super-hamiltonian constraint, i.e. a choice of a particular 3+1 splitting, is done by fixing the momentum π_ϕ conjugate to the conformal factor. This shows the dominant role of the conformal 3-geometries in the determination of the physical degrees of freedom, just as in the Lichnerowicz-York conformal approach.

[27] If $\tilde \phi[r_{\bar a}, \pi_{\bar a}, \xi^r, \pi_\phi]$ is the solution of the Lichnerowicz equation, then $Q_{\mathcal{H}} = \phi - \tilde \phi \approx 0$. Other forms of this canonical transformation should correspond to the extension of the York map [62] to asymptotically flat space-times: in this case the momentum conjugate to the conformal factor would be just York time and one could add the maximal slicing condition as a gauge fixing. Again, however, nobody has been able so far to build a York map explicitly.

construct the *quasi-Shanmugadhasan* transformation. On the other hand, such transformation has the remarkable property that, in the *special gauge* $\pi_\phi(\tau,\vec{\sigma})\approx 0$, the variables $r_{\bar a}$, $\pi_{\bar a}$ form a canonical basis of off-shell DO for the gravitational field *even if* the solution of the Lichnerowicz equation is not known.

The four gauge fixings to the secondary constraints, when written in the quasi-Shanmugadhasan canonical basis, have the following meaning:

i) the three gauge fixings for the parameters ξ^r of the spatial passive diffeomorphisms generated by the super-momentum constraints correspond to the choice of a system of 3-coordinates on Σ_τ[28]. The time constancy of these gauge fixings generates the gauge fixings for the shift functions N_r while the time constancy of the latter leads to the fixation of the Dirac multipliers $\lambda^{\vec{N}}_r$;

ii) the gauge fixing to the super-hamiltonian constraint determines π_ϕ: it is a fixation of the form of Σ_τ and amounts to the choice of one particular 3+1 splitting of M^4. Since the time constancy of the gauge fixing on π_ϕ determines the gauge fixing for the lapse function N (and then of the Dirac multiplier λ_N), it follows a connection with the choice of the standard of local proper time.

All this entails that, after such a fixation of the gauge G, the functional form of the DO in terms of the original variables becomes gauge-dependent. At this point it is convenient to denote them as $r^G_{\bar a}$, $\pi^G_{\bar a}$.

In conclusion, a representative of a *Hamiltonian kinematical or off-shell gravitational field*, in a given gauge equivalence class, is parametrized by $r_{\bar a}$, $\pi_{\bar a}$ and is an element of a *conformal gauge orbit* (it contains all the 3-metrics in a conformal 3-geometry) spanned by the gauge variables ξ^r, π_ϕ, N, N_r. Therefore, according to the gauge interpretation based on constraint theory, a *Hamiltonian kinematical or off-shell gravitational field* is an equivalence class of 4-metrics modulo the Hamiltonian group of gauge transformations, which contains a well defined conformal 3-geometry. Clearly, this is a consequence of the different invariance properties of the ADM and Hilbert actions, even if they generate the same equations of motion.

Recall that the evolution is parametrized by the mathematical parameter τ of the adapted coordinate system $(\tau,\vec{\sigma})$ on M^4, which labels the surfaces Σ_τ. As shown in Ref. [44], the Hamiltonian ruling the evolution is the *weak ADM energy* [63] (the volume form), which, in

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[28]Since the diffeomorphism group has no canonical identity, this gauge fixing has to be done in the following way. We choose a 3-coordinate system by choosing a parametrization of the six components ${}^3g_{rs}(\tau,\vec{\sigma})$ of the 3-metric in terms of *only three* independent functions. This amounts to fix the three functional degrees of freedom associated with the diffeomorphism parameters $\xi^r(\tau,\vec{\sigma})$. For instance, a 3-orthogonal coordinate system is identified by ${}^3g_{rs}(\tau,\vec{\sigma})=0$ for $r\neq s$ and ${}^3g_{rr}=\phi^2\, exp(\sum_{\bar a=1}^2\gamma_{r\bar a}r_{\bar a})$. Then, we impose the gauge fixing constraints $\xi^r(\tau,\vec{\sigma})-\sigma^r\approx 0$ as a way of identifying this system of 3-coordinates with a conventional origin of the diffeomorphism group manifold.

every completely fixed gauge, is a functional of only the DO of that gauge. As shown by DeWitt [64], this is a consequence of the fact that *in non-compact space-times the weakly vanishing ADM Dirac Hamiltonian (3.3) has to be modified with a suitable surface term in order to have functional derivatives, Poisson brackets and Hamilton equations mathematically well defined mathematically.*

Finally, let us stress the important fact that the Shanmugadhasan canonical transformation is a *highly non-local* transformation. Since it is not known how to build a global atlas of coordinate charts for the group manifold of diffeomorphism groups, it is not known either how to express the ξ^r's, π_ϕ and the DO in terms of the original ADM canonical variables.[29].

[29]This should be compared to the Yang-Mills theory in case of a trivial principal bundle, where the corresponding variables are defined by a path integral over the original canonical variables [65,50,51].

IV. PHYSICAL INDIVIDUATION OF SPACE-TIME POINTS BY MEANS OF GAUGE FIXINGS TO BERGMANN-KOMAR INTRINSIC COORDINATES.

It's time to pursue our main task, namely the clarification of the physical individuality of space-time point-events in general relativity.

Let us begin by recalling again that the ADM formulation assumes the existence of a mathematical 4-manifold, the space-time M^4, admitting 3+1 splittings with space-like leaves $\Sigma_\tau \approx R^3$. All fields (also matter fields when present) depend on Σ_τ-adapted coordinates $(\tau, \vec{\sigma})$ for M^4. As emphasized in the Introduction, unlike the case of special relativity, the *mathematical points* of M^4 have no intrinsic physical meaning, a circumstance that originates the *Hole* phenomenology. This has the effect that, as stressed in particular by Stachel [23], the space-time manifold points must be *physically* individuated *before* space-time itself acquires a physical meaning. In the case of general relativity there is *no non-dynamical individuating field* like the distribution of rods and clocks in Minkowsky space-time, that can be specified independently of the dynamical fields, in particular independently of the metric. Thus, Stachel claimed that, for the pure gravitational solutions, *the metric itself plays the role of individuating field* and that this role should be implemented by four invariant functionals of the metric; however, he didn't pursue this program further. We must insist again that a crucial component of the individuation issue is the inextricable entanglement of the Hole Argument with the initial value problem. Now, however, we have at our disposal the right framework for dealing with the initial value problem, so our main task should be to put all things together and develop Stachel's suggestion. Finally, another fundamental tool at our disposal is the clarification we gained in Section II concerning the relation between active diffeomorphisms in their passive view and the dynamical gauge symmetries of Einstein's equations in the Hamiltonian approach.

We are then ready to move forward by conjoining Stachel's suggestion with the proposal advanced by Bergmann and Komar [1] that, in the absence of matter fields, the values of four invariant scalar fields built from the contractions of the Weyl tensor (actually its eigenvalues) can be used to build *intrinsic pseudo-coordinates*[30].

The four invariant scalar eigenvalues $\Lambda^{(k)}_W(\tau, \vec{\sigma})$, $k = 1, .., 4$, of the Weyl tensor, written in Petrov compressed notations, are

$$
\begin{aligned}
\Lambda^{(1)}_W &= Tr\,({}^4C\,{}^4g\,{}^4C\,{}^4g),\\
\Lambda^{(2)}_W &= Tr\,({}^4C\,{}^4g\,{}^4C\,{}^4\epsilon),\\
\Lambda^{(3)}_W &= Tr\,({}^4C\,{}^4g\,{}^4C\,{}^4g\,{}^4C\,{}^4g),\\
\Lambda^{(4)}_W &= Tr\,({}^4C\,{}^4g\,{}^4C\,{}^4g\,{}^4C\,{}^4\epsilon),
\end{aligned}
\tag{4.1}
$$

where ${}^4C$ is the Weyl tensor, ${}^4g$ the metric, and ${}^4\epsilon$ the Levi-Civita totally antisymmetric tensor.

Bergmann and Komar then propose that we build a set of (off-shell) *intrinsic coordinates* for the point-events of space-time as four suitable functions of the $\Lambda^{(k)}_W$'s,

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[30] As shown in Ref. [66] in general space-times with matter there are 14 algebraically independent curvature scalars for M^4.

$$\bar{\sigma}^{\bar{A}}(\sigma) = F^{\bar{A}}[\Lambda_W^{(k)}[^4g(\sigma), \partial^4 g(\sigma)]], \, (\bar{A} = 1, 2, ..., 4). \tag{4.2}$$

Indeed, under the hypothesis of no space-time symmetries,[31] we would be tempted (like Stachel) to use the $F^{\bar{A}}[\Lambda_W^{(k)}]$ as individuating fields to *label the points of space-time*, at least locally.[32]

Of course, since they are invariant functionals, the $F^{\bar{A}}[\Lambda_W^{(k)}]$'s are quantities invariant under passive diffeomorphisms (PDIQ), therefore, as such, they do not define a coordinate chart for the atlas of the mathematical Riemannian 4-manifold M^4 in the usual sense (hence the name of *pseudo-coordinates* and the superior bar we used in $F^{\bar{A}}$). Moreover, the tetradic 4-metric which can be built by means of the intrinsic pseudo-coordinates (see the next Section) is a formal object invariant under passive diffeomorphisms that does not satisfy Einstein's equations (but possibly much more complex derived equations). Therefore, the action of active diffeomorphisms on the tetradic metric is not *directly* connected to the Hole argument. All this leads to the conclusion that the proposal advanced by Bergmann [2] ("we might then identify a *world point* (location-plus-instant-in-time) by the values assumed by (the four intrinsic pseudo-coordinates)") to the effect of individuating point-events in terms of *intrinsic pseudo-coordinates* is not - as it stands - *physically viable* in a tractable way. This is not the final verdict, however, and we must find a dynamical bridge between the intrinsic pseudo-coordinates and the ordinary 4-coordinate systems which provide the primary identification of the points of the mathematical manifold.

Our procedure starts when we recall that, within the Hamiltonian approach, Bergmann and Komar [1] proved the fundamental result that the Weyl eigenvalues $\Lambda_W^{(A)}$, once re-

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[31]Our attempt to use intrinsic coordinates to provide a physical individuation of point-events would *prima facie* fail in the presence of symmetries (with or without matter), when the $F^{\bar{A}}[\Lambda_W^{(k)}[^4g(\sigma), \partial^4 g(\sigma)]]$ become degenerate. This objection was originally raised by Norton [12] as a critique to manifold-plus-further-structure (MPFS) substantivalism (according to which the points of the manifold, conjoined with additional local structure such as the metric field, can be considered physically real; see for instance [67]). Several responses are possible. First, although most of the *known* exact solutions of the Einstein equations admit one or more symmetries, these mathematical models are very idealized and simplified; in a realistic situation (for instance, even with two masses) space-time is filled with the excitations of the gravitational degrees of freedom, and admits no symmetries at all. A case study is furnished by the non-symmetric and non-singular space-times of Christodoulou-Klainermann [36]. Second, the parameters of the symmetry transformations can be used as supplementary individuating fields, since, as noticed by Komar [30] and Stachel [23] they also depend upon metric field, through its isometries. To this move it has been objected [24] that these parameters are purely mathematical artifacts, but a simple rejoinder is that the symmetric models too are mathematical artifacts. Third, and most important, in our analysis of the physical individuation of points we are arguing a question of principle, and therefore we must consider *generic* solutions of the Einstein equations rather than the null-measure set of solutions with symmetries.

[32]Problems might arise if we try to extend the labels to the entire space-time: for instance, the coordinates might turn out to be multi-valued.

expressed as functionals of the Dirac (i.e. ADM) canonical variables, *do not depend on the lapse and shift functions* but only on the 3-metric and its conjugate canonical momentum, $\Lambda_W^{(k)}[{}^4g(\tau,\vec{\sigma}),\partial^4 g(\tau,\vec{\sigma})] = \tilde{\Lambda}_W^{(k)}[{}^3g(\tau,\vec{\sigma}),{}^3\Pi(\tau,\vec{\sigma})]$. This result is crucial since it entails that just the *intrinsic pseudo-coordinates* $\bar{\sigma}^{\bar{A}}$ can be exploited as natural and peculiar *coordinate gauge conditions* in the canonical reduction procedure.

Taking into account the results of Section III, we know that, in a completely fixed (either *off-* or *on-shell*) gauge, both the four intrinsic *pseudo-coordinates* and the ten *tetradic* components of the metric field (see Eq.(5.2) of the next Section) become gauge dependent functions of the four DO of that gauge. For the Weyl scalars in particular we can write:

$$\Lambda_W^{(k)}(\tau,\vec{\sigma})|_G = \tilde{\Lambda}_W^{(k)}[{}^3g(\tau,\vec{\sigma}),{}^3\Pi(\tau,\vec{\sigma})]|_G = \Lambda_G^{(k)}[r^G_{\bar{a}}(\tau,\vec{\sigma}),\pi^G_{\bar{a}}(\tau,\vec{\sigma})]. \tag{4.3}$$

where $|_G$ denotes the specific gauge (see footnote 25). Conversely, by the inverse function theorem, in each gauge, the DO of that gauge can be expressed as functions of the 4 eigenvalues restricted to that gauge: $\Lambda_W^{(k)}(\tau,\vec{\sigma})|_G$.

Our program is implemented in the following way: after having selected a *completely arbitrary mathematical* coordinate system $\sigma^A \equiv [\tau,\sigma^a]$ adapted to the Σ_τ surfaces, we choose *as physical individuating fields* four suitable functions $F^{\bar{A}}[\Lambda_W^{(k)}(\tau,\vec{\sigma})]$, and express them as functionals $\tilde{F}^{\bar{A}}$ of the ADM variables

$$F^{\bar{A}}[\Lambda_W^{(k)}(\tau,\vec{\sigma})] = F^{\bar{A}}[\tilde{\Lambda}_W^{(k)}[{}^3g(\tau,\vec{\sigma}),{}^3\Pi(\tau,\vec{\sigma})]] = \tilde{F}^{\bar{A}}[{}^3g(\tau,\vec{\sigma}),{}^3\Pi(\tau,\vec{\sigma})]. \tag{4.4}$$

The space-time points, *mathematically individuated* by the quadruples of real numbers σ^A, become now *physically individuated point-events* through the imposition of the following gauge fixings to the four secondary constraints

$$\bar{\chi}^A(\tau,\vec{\sigma}) \stackrel{def}{=} \sigma^A - \bar{\sigma}^{\bar{A}}(\tau,\vec{\sigma}) = \sigma^A - F^{\bar{A}}\Big[\tilde{\Lambda}_W^{(k)}[{}^3g(\tau,\vec{\sigma}),{}^3\Pi(\tau,\vec{\sigma})]\Big] \approx 0. \tag{4.5}$$

Then, following the standard procedure we end with a completely fixed Hamiltonian gauge, say G. This will be a correct gauge fixing provided the functions $F^{\bar{A}}[\Lambda_W^{(k)}(\tau,\vec{\sigma})]$ are chosen so that the $\bar{\chi}^A(\tau,\vec{\sigma})$'s satisfy the *orbit conditions*

$$det\,|\{\bar{\chi}^A(\tau,\vec{\sigma}),\tilde{\mathcal{H}}^B(\tau,\vec{\sigma}^{\,\prime})\}| \neq 0, \tag{4.6}$$

where $\tilde{\mathcal{H}}^B(\tau,\vec{\sigma}) = \Big(\tilde{\mathcal{H}}(\tau,\vec{\sigma});{}^3\tilde{\mathcal{H}}^r(\tau,\vec{\sigma})\Big) \approx 0$ are the super-hamiltonian and super-momentum constraints of Eqs.(3.2). These conditions enforce the Lorentz signature on Eq.(4.5), namely the requirement that $F^{\bar{\tau}}$ be a *time* variable, and imply that *the* $F^{\bar{A}}$*'s are not DO.*

The above gauge fixings allow in turn the determination of the four Hamiltonian gauge variables $\xi^r(\tau,\vec{\sigma})$, $\pi_\phi(\tau,\vec{\sigma})$ of Eqs.(3.4). Then, their time constancy induces the further gauge fixings $\bar{\psi}^A(\tau,\vec{\sigma}) \approx 0$ for the determination of the remaining gauge variables, i.e., the lapse and shift functions in terms of the DO in that gauge as

$$\begin{aligned}
\dot{\bar{\chi}}^A(\tau,\vec{\sigma}) &= \frac{\partial \bar{\chi}^A(\tau,\vec{\sigma})}{\partial \tau} + \{\bar{\sigma}^{\bar{A}}(\tau,\vec{\sigma}),\bar{H}_D\} = \delta^{A\tau} + \\
&+ \int d^3\sigma_1 \Big[N(\tau,\vec{\sigma}_1)\,\{\sigma^{\bar{A}}(\tau,\vec{\sigma}),\mathcal{H}(\tau,\vec{\sigma}_1)\} + N_r(\tau,\vec{\sigma}_1)\,\{\sigma^{\bar{A}}(\tau,\vec{\sigma}),\mathcal{H}^r(\tau,\vec{\sigma}_1)\}\Big] = \\
&= \bar{\psi}^A(\tau,\vec{\sigma}) \approx 0.
\end{aligned} \tag{4.7}$$

Finally, $\dot{\bar{\psi}}^A(\tau, \vec{\sigma}) \approx 0$ determines the Dirac multipliers $\lambda^A(\tau, \vec{\sigma})$.

In conclusion, the gauge fixings (4.5) (*which break general covariance*) constitute the crucial bridge that transforms the *intrinsic pseudo-coordinates* into *true physical individuating coordinates.*

As a matter of fact, after going to Dirac brackets, we enforce the point-events individuation in the form of the *identity*

$$\sigma^A \equiv \bar{\sigma}^{\bar{A}} = \tilde{F}^{\bar{A}}_G[r^G_{\bar{a}}(\tau, \vec{\sigma}), \pi^G_{\bar{a}}(\tau, \vec{\sigma})] = F^{\bar{A}}[\Lambda^{(k)}_W(\tau, \vec{\sigma})]|_G. \tag{4.8}$$

In this *physical 4-coordinate grid*, the 4-metric, as well as other fundamental physical entities, like e.g. the space-time interval ds^2 with its associated causal structure, and the lapse and shift functions, depend entirely on the DO in that gauge. The same is true, in particular, for the solutions of the eikonal equation [36] ${}^4g^{AB}(\sigma^D)\, \frac{\partial U(\sigma^D)}{\partial \sigma^A}\, \frac{\partial U(\sigma^D)}{\partial \sigma^B} = 0$, which define generalized wave fronts and, therefore, through the envelope of the null surfaces $U(\sigma^D) = const.$ at a point, the light cone at that point.

Let us stress that, according to the previous Sections, only on the solutions of Einstein's equations the completely fixed gauge G is equivalent to the fixation of a definite 4-coordinate system σ^A_G. Our gauge fixing (4.5) ensures that *on-shell* we get $\sigma^A = \sigma^A_G$. In this way we get a physical 4-coordinate grid on the mathematical 4-manifold M^4 dynamically determined by tensors over M^4 with a rule which is invariant under ${}_PDiff\, M^4$ but such that the functional form of the map $\sigma^A \mapsto physical\ 4 - coordinates$ depends on the complete chosen gauge G: we see that what is usually called the *local point of view* [68] (see later on) is justified a posteriori in every completely fixed gauge.

Summarizing, the effect of the whole procedure is that *the values of the DO, whose dependence on space (and on parameter time) is indexed by the chosen coordinates* $(\tau, \vec{\sigma})$, *reproduces precisely such* $(\tau, \vec{\sigma})$ *as the Bergmann–Komar intrinsic coordinates in the chosen gauge* G. In this way *mathematical points have become physical individuated point-events* by means of the highly non-local structure of the DO. If we read the identity (4.8) as $\sigma^A \equiv f^{\bar{A}}_G(r^G_{\bar{a}}, \pi^G_{\bar{a}})$, we see that each coordinate system σ^A is determined on-shell by the values of the 4 canonical degrees of freedom of the gravitational field in that gauge. This is tantamount to claiming that *the physical role and content of the gravitational field in absence of matter is just the very identification of the points of Einstein space-times into physical point-events by means of its four independent phase space degrees of freedom.* The existence of physical point-events in general relativity appears here as a synonym of the existence of the DO, i.e. of the true physical degrees of freedom of the gravitational field.

As said in the Introduction, the addition of matter does not change this ontological conclusion, because we can continue to use the gauge fixing (4.5). However, matter changes the Weyl tensor through Einstein's equations and contributes to the separation of gauge variables from DO in the quasi-Shanmugadhasan canonical transformation through the presence of its own DO. In this case we have Dirac observables both for the gravitational field and for the matter fields, but *the former are modified in their functional form* with respect to the vacuum case by the presence of matter. Since the gravitational Dirac observables will

still provide the individuating fields for point-events according to our procedure, *matter will come to influence the very physical individuation of points.*

We have seen that, once the orbit conditions are satisfied, the Bergmann-Komar *intrinsic pseudo-coordinates* $F^{\bar{A}}[\tilde{\Lambda}_W^{(k)}[{}^3g(\tau,\vec{\sigma}), {}^3\Pi(\tau,\vec{\sigma})]|_G$ become just the *individuating fields* Stachel was looking for. Indeed, by construction, the intrinsic pseudo-coordinates are both PDIQ and also *numerically invariant* under the drag along induced by active diffeomorphisms (in the notations of the Introduction we have $[\phi^* F^{\bar{A}}](p) \equiv [F^{\bar{A}\,\prime}](p) = [F^{\bar{A}}](\phi^{-1}\cdot p)$), a fact that is also essential for maintaining a connection to the *Hole Argument*.

A better understanding of the real connection between Stachel point of view and ours can be achieved by exploiting Bergmann-Komar's group of passive transformations Q discussed in Section II. We can argue in the following way. Given a 4-coordinate system σ^A, the passive view of each active diffeomorphism ϕ defines a new 4-coordinate system σ_ϕ^A (drag-along coordinates produced by a generalized Bergmann-Komar transformation (2.3)). This means that there will be two functions $F^{\bar{A}}$ and $F_\phi^{\bar{A}}$ realizing these two coordinates systems through the gauge fixings

$$\begin{aligned} &\sigma^A - F^{\bar{A}}[\Lambda_W^{(k)}(\sigma)] \approx 0, \\ &\sigma_\phi^A - F_\phi^{\bar{A}}[\Lambda_W^{(k)}(\sigma_\phi)] \approx 0, \end{aligned} \tag{4.9}$$

It is explicitly seen in this way that the functional freedom in the choice of the four functions $F^{\bar{A}}$ allows to cover all those coordinates charts σ^A in the atlas of the mathematical space-time M^4 which are adapted to any allowed 3+1 splitting. By using gauge fixing constraints more general than those in Eq.(4.5) (like the standard gauge fixings used in ADM metric gravity) we can reach all the 4-coordinates systems of M^4. Here, however we wanted to restrict to the class of gauge fixings (4.5) for the sake of clarifying the interpretational issues.

Let us conclude by noting that the gauge fixings (4.5), (4.7) induce a *coordinate-dependent non-commutative Poisson bracket structure* upon the *physical point-events* of space-time by means of the associated Dirac brackets implying Eqs.(4.8). More exactly, on-shell, each coordinate system gets a well defined non-commutative structure determined by the associated functions $\tilde{F}_G^{\bar{A}}(r_{\bar{a}}^G, \pi_{\bar{a}}^G)$, for which we have $\{\tilde{F}_G^{\bar{A}}(r_{\bar{a}}^G(\tau,\vec{\sigma}), \pi_{\bar{a}}^G(\tau,\vec{\sigma})), \tilde{F}_G^{\bar{B}}(r_{\bar{a}}^G(\tau,\vec{\sigma}_1), \pi_{\bar{a}}^G(\tau,\vec{\sigma}_1))\}^* \neq 0$. The meaning of this structure in view of quantization is worth investigating (see the Concluding Survey).

V. A DIGRESSION ON BERGMANN OBSERVABLES: THE MAIN CONJECTURE AND THE ISSUE OF THE OBJECTIVITY OF CHANGE.

This Section is devoted to some crucial aspects of the definition of *observable* in general relativity. While, for instance in astrophysics, *matter observables* are usually defined as *tetradic quantities* evaluated with respect to the tetrads of a time-like observer so that they are obviously invariant under ${}_PDiff\, M^4$ (PDIQ), the definition of the notion of observable for the gravitational field without matter faces a dilemma. Two fundamental definitions of *observable* have been proposed in the literature.

1) The *off-shell and on-shell Hamiltonian non-local Dirac observables* (DO) [33] which, by construction, satisfy hyperbolic Hamilton equations of motion and are, therefore, deterministically *predictable*. In general, as already said, they are neither tensorial quantities nor invariant under ${}_PDiff\, M^4$ (PDIQ).

2) The *configurational Bergmann observables* (*BO*) [2]: they are quantities defined on M^4 *which not only are independent of the choice of the coordinates*, i.e. they are either scalars or invariants[34] under ${}_PDiff\, M^4$ (PDIQ), but are also "*uniquely predictable from the initial data*". An equivalent, but according to Bergmann more useful, definition of a (PIDQ) BO, is "*a quantity that is invariant under a coordinate transformation that leaves the initial data unchanged*".

Let us note, first of all, that PDIQ's are *not* in general DO, because they may also depend on the eight gauge variables N, N_r, ξ^r, π_ϕ. Thus most, if not all, of the curvature scalars are gauge dependent quantities at least at the kinematical off-shell level. For example, each 3-metric in the conformal gauge orbit has a different 3-Riemann tensor and different 3-curvature scalars. Since 4-tensors and 4-curvature scalars depend: i) on the lapse and shift functions (and their gradients); ii) on π_ϕ, both implicitly and explicitly through the solution of the Lichnerowicz equation (and this affects the 3-curvature scalars), most of these objects

[33] For other approaches to the observables of general relativity see Refs. [69]: the *perennials* introduced in this Reference are essentially our DO. See Ref. [70] for the difficulties in observing *perennials* experimentally at the classical and quantum levels as well as for their quantization. See Ref. [71] about the non-existence of observables built as spatial integrals of local functions of Cauchy data and their first derivatives, in the case of vacuum gravitational field in a closed universe. Also, Rovelli's evolving constants of motion and partial observables [72] are related with DO; however, the holonomy loops used in loop quantum gravity [73] are PDIQ but not DO. On the other hand, even recently Ashtekar [74] noted that "The issue of diffeomorphism invariant observables and practical methods of computing their properties" is one among the relevant challenges.

[34] In Ref. [29] Bergmann defines: i) a *scalar* as a local field variable which retains its numerical value at the same world point under coordinate transformations (passive diffeomorphisms), $\varphi^{'}(x^{'}) = \varphi(x)$; ii) an *invariant* I as a functional of the given fields which has been constructed so that if we substitute the coordinate transforms of the field variables into the argument of I instead of the originally given field variables, then the numerical value of I remains unchanged.

are in general gauge dependent variables from the Hamiltonian point of view, at least at the off-shell kinematical level. The simplest relevant off-shell scalars with respect to ${}_P Diff\, M^4$, which exhibit such gauge dependence, are the bilinears ${}^4R_{\mu\nu\rho\sigma}\, {}^4R^{\mu\nu\rho\sigma}$, ${}^4R_{\mu\nu\rho\sigma}\, \epsilon^{\mu\nu\alpha\beta}\, {}^4R_{\alpha\beta}{}^{\rho\sigma}$ and the *four eigenvalues of the Weyl tensor* exploited in Section IV. What said here *does hold, in particular, for the line element ds^2 and, therefore, for the causal structure.*

On the other hand, BO are those special PDIQ which are simultaneously *predictable*. Yet, the crucial question is now "*what does it precisely mean to be predictable within the configurational Lagrangian framework* ?". Bergmann, gave in fact a third definition of BO or, better, a *third part* of the original definition, as "a dynamical variable that (from the Hamiltonian point of view) has vanishing Poisson brackets with all the constraints", i.e., essentially, is also a DO. This means that Bergmann thought, though only implicitly and without proof, that *predictability* implies that a BO must *also* be *projectable to phase space to a special subset of DO that are also PDIQ*.

The unresolved multiplicity of Bergmann's definitions leads to an entangled net of problems. First of all, as shown at length in Ref. [28], in order to tackle the Cauchy problem at the Lagrangian configuration level [35] one has firstly to disentangle the Lagrangian constraints from Einstein's equations, then to take into account the Bianchi identities, and finally to write down a system of hyperbolic equations. As a matter of fact one has to mimic the Hamiltonian approach, but with the additional burden of lacking an algorithm for selecting those predictable configurational field variables whose Hamiltonian counterparts are just the DO. The only thing one might do is to adopt an inverse Legendre transformation, to be performed after a Shanmugadhasan canonical transformation characterizing a possible complete set of DO. Yet, this corresponds just to the inverse of Bergmann's statement that the BO are projectable to special (PDIQ) DO. In conclusion *Lagrangian configurational predictability must be equivalent to the statement of off-shell gauge invariance*. The moral is that the complexity of the issue should warn against any unconstrained utilization of geometric intuitions in dealing with the initial value problem of general relativity.

This Hamiltonian predictability of BO entails in turn that only *four functionally independent BO can exist for the vacuum gravitational field*, since the latter has only two pairs of conjugate independent degrees of freedom. Let us see now why Bergmann's multiple definition of BO raises additional subtle problems.

Bergmann himself proposed a constructive procedure for the BO. This is essentially based on his re-interpretation of Einstein's *coincidence argument* in terms of the individuation of space-time points as point-events by using *intrinsic pseudo-coordinates*. In his - already quoted - words [2]: "we might then identify a *world point* (location-plus-instant-in-time) by the values assumed by (the four intrinsic pseudo-coordinates) and ask for the value, there and then, of a fifth field". As an instantiation of this procedure, Bergmann refers to Komar's [30] pseudo-tensorial transformation of the 4-metric tensor to the *intrinsic pseudo-coordinate* system [$\sigma^A = \sigma^A(\bar\sigma^{\bar A})$ is the inversion of Eqs.(4.2)]

$$ {}^4\bar g_{\bar A\bar B}(\bar\sigma^{\bar C}) = \frac{\partial \sigma^C}{\partial \bar\sigma^{\bar A}}\, \frac{\partial \sigma^D}{\partial \bar\sigma^{\bar B}}\, {}^4g_{CD}(\sigma). \tag{5.1} $$

[35] In the theory of systems of partial differential equations this is done in a *passive* way in a given coordinate system and then extended to all coordinate systems

The ${}^4\bar{g}_{\bar{A}\bar{B}}$ represent *ten invariant scalar* (PDIQ or *tetradic*) *components* of the metric; of course, they are not all independent since must satisfy eight functional restrictions following from Einstein's equations.

Now, Bergmann *claims* that the ten components ${}^4\bar{g}_{\bar{A}\bar{B}}(\bar{\sigma}^{\bar{C}})$ are a complete, but non-minimal, set of BO. *This claim, however, cannot be true.* As already pointed out, since BO are predictable they must in fact be equivalent to (PDIQ) DO so that, for the vacuum gravitational field, exactly four functions at most, out of the ten components ${}^4\bar{g}_{\bar{A}\bar{B}}(\bar{\sigma}^{\bar{C}})$, can be simultaneously BO and DO, while the remaining components must be *non-predictable* PDIQ, counterparts of ordinary Hamiltonian gauge variables.

On the other hand, as shown in Section IV, the four independent degrees of freedom of the pure vacuum gravitational field, even for Bergmann, *have allegedly already been exploited for the individuation of point-events*. Besides, as Bergmann explicitly asserts in his purely *passive* interpretation, the PDIQ ${}^4\bar{g}_{\bar{A}\bar{B}}(\bar{\sigma}^{\bar{C}})$ identify on-shell a 4-geometry, i.e. an equivalence class in ${}^4Geom = {}^4Riem/_P Diff\, M^4$. Furthermore, as shown in Section II, Eq.(2.7), the identification of the same 4-geometry starting from active diffeomorphisms can be done by using their passive re-formulation (the group Q). Finally, let us remark that Bergmann's intention to exploit first the intrinsic pseudo-coordinates and then "ask for the value, there and then, of a fifth field" makes sense only if such "*fifth field*" is a *matter field*. Asking the question for purely gravitational quantities like ${}^4\bar{g}_{\bar{A}\bar{B}}(\bar{\sigma}^{\bar{C}})$ would be at least tautological since, as we have seen, only four of them can be independent and have already been exploited. If the individuation procedure is intended to be effective, it would make little sense to assert that point-events have such and such values in terms of point-events.

But now, Bergmann's incorrect claim is relevant also to another interesting quandary. Indeed, Bergmann's *main* configurational notion of *observable* and its implications are accepted as they stand in the already quoted paper of John Earmann, Ref. [3]. In particular Earman notes that the intrinsic coordinates $[\bar{\sigma}^{\bar{C}}]$ can be used to support Bergmann's observables and says "one can speak of the event of the metric - components - ${}^4\bar{g}_{\bar{A}\bar{B}}(\bar{\sigma}^{\bar{C}})$ - having - such - and - such - values - in - the - coordinate - system - $\{\bar{\sigma}^{\bar{C}}\}$ - at - the - location - where - the - $\bar{\sigma}^{\bar{C}}$ - take - on - values - such - and - so" and (aptly) calls such an item a *Komar event*, adding moreover that "the fact that a given Komar event occurs (or fails to occur) is an observable matter in Bergmann's sense, albeit in an abstract sense because how the occurrence of a Komar event is to be observed/measured is an unresolved issue" [36]. Earman's principal aim, however, is to exploit Bergmann's definition of BO to show that "it implies that there is no physical change, i.e., no change in the observable quantities, at least not for those quantities that are constructible in the most straightforward way from the materials at hand". Although we are not committed here to object to what Earman calls "modern Mc-Taggart argument" about change, we are obliged to take issue against Earman's radical *universal no-change argument* because, if sound, it would contradict the substance of Bergmann's definition of predictability and would jeopardize the relation between BO and DO which is fundamental to our program.

In order to scrutinize this point, let us resume, for the sake of clarity, the essential basic ingredients of the present discussion.

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[36]For our point of view about this issue, see Section VII

One: the equations of motion derived from Einstein-Hilbert Action and those derived from ADM Action have *exactly the same physical content*: the ADM Lagrangian leads, through the Legendre transformation, to equations equivalent to Einstein's equations. *Two*: Hamiltonian predictability must, therefore, be equivalent to Lagrangian predictability: specification of the latter, however, is awkward. *Three*: the only functionally independent Hamiltonian predictable quantities for the vacuum gravitational field, are 4 DO. *Four*: by inverse Legendre transformation, every DO has *a Lagrangian predictable counterpart*. Then, a priori, one among the following three possibilities might be true: i) all the existing BO must also be DO; this means however that only four functionally independent BO can exist; ii) some of the existing BO are also DO while other are not; iii) no one of the existing BO is also a DO. Possibilities ii) and iii) entail that Bergmann's multiple definition (that including the third part) of BO is inconsistent, so that no BO satisfying such multiple definition would exist. Yet, the third part of Bergmann definition is essential for the overall meaning of it since no Lagrangian definition of predictability *independent of its Hamiltonian counterpart* can exist because of *Two*. Thus cases ii) and iii) imply inconsistency of the very concept of Bergmann's observabiity. Of course, it could be that even i) is false since, after all, Bergmann did not prove the self-consistency of his multiple definition: but this would mean that no Lagrangian predictable quantity can exist which is simultaneously a PDIQ. Here, we are *assuming* that Bergmann's multiple definition is consistent and that i) is true. We will formalize this assumption into a definite *constructive conjecture* later on in this Section.

Let us take up again the discussion about the reality of change. As already noted, the discussion in terms of BO in the language of *Komar events* (or *coincidences*) must be restricted to the properties of matter fields because, consistently with the multiple Bergmann's definition, only four of the BO can be purely gravitational in nature. And, if these latter have already been exploited for the individuation procedure, it would again make little sense to ask whether point-events do or do not change. Therefore, let us consider Earman's argument by examining his interpretation of predictability and the consequent implications for a BO, say $B(p)$, $p \in M^4$, which, besides depending on the 4-metric and its derivatives up to some finite order, also depends on matter variables, and is of course a PDIQ. In order to simplify the argument, Earman concentrates on the special case of the vacuum solutions to the Einstein's field equations, asserting however that the argument easily generalizes to non-vacuum solutions. Since we have already excluded the case of vacuum solutions, let us take for granted that this generalization is sound. Earman argues essentially in the following way: 1) There are existence and uniqueness proofs for the initial value problem of Einstein's equations, which show that for appropriate initial data associated to a three manifold $\Sigma_o \subset M^4$, there is a unique *up to diffeomorphism* (obviously to be intended *active*) [37] maximal development for which Σ_o is a Cauchy surface; 2) By definition, a BO is a PDIQ whose value $B(p)$ at some point p in the future of Σ_o is predictable from initial data on Σ_o.

[37]Note that Earman deliberately deviates here from the purely *passive viewpoint* of Bergmann (and of the standard Cauchy problem for partial differential equations) by resorting to *active* diffeomorphisms in place of the *coordinate transformations that leave the initial data unchanged* or, possibly, in place of their extension in terms of the passive re-interpretation of active diffeomorphisms (Q group).

If $\phi : M^4 \mapsto M^4$ is an *active* diffeomorphism that leaves Σ_o and its past fixed, the point p will be sent to the point $p^{'} = \phi \cdot p$. Then, the general covariance of Einstein's equations, *conjoined with predictability*, is interpreted to imply $B(p) = B(\phi \cdot p)$. This result, together with the definition $B'(p) = B(\phi^{-1} \cdot p)$ of the drag along of B under the active diffeomorphism ϕ, entails $\phi^* B = B$ for a BO. In conclusion, since Σ_o is arbitrary, a matter BO should be *constant everywhere* in M^4.

It is clear that this conclusion cannot hold true for any matter dependent BO that is projectable to the DO of the gravitational field *cum* matter, if only for the fact that such BO are in fact ruled by the weak *ADM energy* which generates real temporal change (see Section III and Eq.(5.3) below). The crucial point in Earman's argument is the assertion that predictability implies $B(p) = B(\phi \cdot p)$. But this does *not* correspond to the property of *off-shell gauge invariance* spelled above as the main qualification of predictable quantities, except of course for the trivial case of quantities everywhere constant. As clarified in Section II, the relations between active diffeomophisms and gauge transformation (which are necessarily involved by the DO) is not straightforward. Precisely, because of the properties of the group Q of Bergmann and Komar, we have to distinguish between the *active* diffeomorphisms in Q that *do belong* to Q_{can} and those that *do not belong* to Q_{can} [38].

It is clear that, lacking projectability to phase space, these latter do not correspond to gauge transformations and have, therefore, nothing to do with Lagrangian *predictability*. Thus, for most *active* diffeomorphisms[39], the conclusion $B(p) = B(\phi \cdot p)$ cannot hold true. This erroneous conclusion seems to be just an instantiation of how misleading may be any loose geometrical and non-algorithmic interpretation of Σ_o as a Cauchy surface within the Lagrangian configuration approach to the initial value problem of general relativity.

Having settled this important point, let us come back to tetradic fields. Besides the tetradic components (5.1) of the 4-metric, we have to take into account the extrinsic curvature tensor ${}^3K^{AB}(\sigma) = \frac{\partial \sigma^A}{\partial x^\mu} \frac{\partial \sigma^B}{\partial x^\nu} {}^3K^{\mu\nu}(x)$. In the coordinates σ^A adapted to Σ_τ, it has the components ${}^3K^{\tau\tau}(\sigma) = {}^3K^{\tau r}(\sigma) = 0$ and ${}^3K^{rs}(\sigma)$ [see Eq.(A6)] and we can rewrite it as

$$ {}^3\bar{K}^{\bar{A}\bar{B}}(\bar{\sigma}^{\bar{C}}) = \frac{\partial \bar{\sigma}^{\bar{A}}}{\partial \sigma^A} \frac{\partial \bar{\sigma}^{\bar{B}}}{\partial \sigma^B} {}^3K^{AB}(\sigma) = \frac{\partial \bar{\sigma}^{\bar{A}}}{\partial \sigma^r} \frac{\partial \bar{\sigma}^{\bar{B}}}{\partial \sigma^s} {}^3K^{rs}(\sigma). \tag{5.2} $$

In this way we get 10 additional scalar (tetradic) quantities (only two of which are independent) replacing ${}^3K^{rs}(\sigma)$ and, therefore, the ADM momenta ${}^3\tilde{\Pi}^{rs}(\sigma)$ of Eqs.(A10).

In each intrinsic coordinate system $\bar{\sigma}^{\bar{A}} = F^{\bar{A}}[\Lambda_W^{(k)}(\sigma)]$, we get consequently the 20 scalar (tetradic) components ${}^3\bar{g}_{\bar{A}\bar{B}}(\bar{\sigma}^{\bar{C}})$ and ${}^3\bar{K}^{\bar{A}\bar{B}}(\bar{\sigma}^{\bar{C}})$ of Eqs.(5.1), (5.2). However, only

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[38]Recall that: i) the intersection $Q_{can} \cap {}_P Diff\, M^4$ identifies the space-time *passive* diffeomorphisms which, respecting the 3+1 splitting of space-time, are *projectable* to $\mathcal{G}_{4\,P}$ in phase space;
ii) the remaining elements of Q_{can} are the *projectable* subset of *active diffeomorphisms* in their passive view.).
iii) the elements of Q which do not belong to Q_{can} are not projectable to phase space at all.

[39]In particular, the special ϕ's considered by Earman

16 out of them are functionally independent because of the 4 scalar intrinsic constraint $\bar{\mathcal{H}}^{\bar{A}}(\bar{\sigma}^{\bar{C}}) = \frac{\partial \bar{\sigma}^{\bar{A}}}{\partial \sigma^A} \mathcal{H}^A(\sigma) \approx 0$ that replace the super-hamiltonian and super-momentum constraints $\mathcal{H}^a(\sigma) = \Big(\mathcal{H}(\sigma); \mathcal{H}^r(\sigma)\Big) \approx 0$.

The various aspects of the discussion given above strongly suggest that, in order to give consistency to Bergmann's unresolved multiple definition of BO and, in particular, to his (strictly speaking unproven) claim [2] of *existence* of DO that are simultaneously (PDIQ) BO, the following *conjecture* should be true:

Main Conjecture: "The Darboux basis whose 16 ADM variables consist of the 8 Hamiltonian gauge variables N, N_r, ξ^r, π_ϕ, the 3 Abelianized constraints $\tilde{\pi}_r^{\vec{\mathcal{H}}} \approx 0$, the conformal factor ϕ (to be determined by the super-hamiltonian constraint) and the (non-tensorial) DO $r_{\bar{a}}$, $\pi_{\bar{a}}$, appearing in the quasi-Shanmugadhasan canonical basis (3.4) *can be replaced* by a Darboux basis whose 16 variables are all PDIQ (or tetradic variables), such that four of them are simultaneous DO and BO, eight vanish because of the first class constraints, and the other 8 are coordinate-independent gauge variables.[40]"

If this conjecture is sound, it would be possible to construct an *intrinsic* Darboux basis of the Shanmugadhasan type (Eq.(3.4)). Then a F-dependent transformation performed off-shell before adding the gauge fixings $\sigma^A - \bar{\sigma}^{\bar{A}}(\sigma) \approx 0$, should exist bringing from the non-tensorial Darboux basis (3.4) to this new *intrinsic* basis. Since the final result would be a representation of the gauge variables as coordinate-independent (PIDQ) gauge variables and of the DO as *Dirac-and-Bergmann observables*, the only F-dependence should consist in the possibility of mixing the PDIQ gauge variables among themselves and in making canonical transformations in the subspace of the Dirac-Bergmann observables.

More precisely, we would have a family of quasi-Shanmugadhasan canonical bases in which all the variables are PDIQ and include 8 PDIQ first class constraints that play the role of momenta. It would be interesting to check the form of the eighth constraint replacing the standard super-hamiltonian constraint. By re-expressing the 4 Weyl eigenvalues in terms of anyone of these PDIQ canonical bases, we could still define a Hamiltonian gauge, namely an on-shell 4-coordinate system and then derive the associated individuation of point-events by means of gauge-fixings of the type (4.5). *Note that this would break general covariance even if the canonical basis is PDIQ*! The only difference with respect to the standard bases would be that, instead of being non-tensorial quantities, both r_a^G, π_a^G and $\tilde{F}_G^{\bar{A}}$ in Eq.(4.8) would be PDIQ.

As anticipated in the Introduction, further strong support to the conjecture comes from Newman-Penrose formalism [38] where the basic tetradic fields are the 20 Weyl and Ricci scalars which are PDIQ by construction . While the vanishing of the Ricci scalars is equivalent to Einstein's equations (and therefore to a scalar form of the super-hamiltonian and super-momentum constraints), the 10 Weyl scalars plus 10 scalars describing the ADM momenta (restricted by the four primary constraints) should lead to the construction of

[40]Note that Bergmann's constructive method based on tetradic 4-metric is not by itself conclusive in this respect !

a Darboux basis spanned only by PDIQ restricted by eight PDIQ first class constraints. Again, a quasi-Shanmugadhasan transformation should produce the Darboux basis of the conjecture. The problem of the phase space re-formulation of Newman-Penrose formalism is now under investigation [75].

A final important logical component of the issue of the objectivity of change is the particular question of *temporal* change. This aspect of the issue is not usually tackled as a sub-case of Earman's *no-universal-change* argument discussed above in terms of BO, so it should be answered separately. We shall restrict our remarks to the objections raised by Belot and Earmann [32] and Earman [3] (see also Refs. [33–35] for the so called *problem of time* in general). According to these authors, the reduced phase space of general relativity is a frozen space without evolution (see in particular the penetrating and thorough discussion about time and change given in Ref. [3]). Belot and Earman draw ontological conclusions about the absence of real (temporal) change in general relativity from the circumstance that, in spatially compact models of general relativity, the Hamiltonian temporal evolution boils down to a mere gauge transformation and is, therefore, physically meaningless. We want to stress, however, that this result *does not apply to all families of Einstein space-times.* In particular, there exist space-times like the Christodoulou-Klainermann space-times [36] we are using in this paper that constitute a *counterexample to the frozen time argument.* They are defined by suitable boundary conditions, are globally hyperbolic, non-compact, and asymptotically flat at spatial infinity. The existence of such meaningful counterexamples entails, of course, that we are not allowed to draw negative *ontological* conclusions in general about the issue of temporal change in general relativity.

More precisely, it is possible to show [44] (see also Section III) that:

1) The imposition of suitable boundary conditions on the fields and the gauge transformations of canonical ADM metric gravity eliminates the super-translations and reduces the asymptotic symmetries at spatial infinity to the asymptotic ADM Poincaré group. The asymptotic implementation of Poincaré group makes possible the general-relativistic definition of angular momentum and the matching of general relativity with particle physics.

2) The boundary conditions of point 1) require that the leaves of the foliations associated with the admissible 3+1 splittings of space-time must tend to Minkowski space-like hyperplanes asymptotically orthogonal to the ADM 4-momentum in a direction-independent way. This property is concretely enforced by using a technique introduced by Dirac [45] for thye selection of space-times admitting asymptotically flat 4-coordinates at spatial infinity. [41]

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[41] Dirac's method brings to an enlargement of ADM canonical metric gravity with non-vanishing ADM Poincaré charges. Such space-times admit preferred asymptotic inertial observers, interpretable as fixed stars (the standard for measuring rotations). Such non-Machian properties allow to merge the standard model of elementary particles in general relativity with all the (gravitational and non-gravitational) fields belonging to the same function space (suitable weighted Sobolev spaces). Besides the existence of a realization of the Poincaré group, only one additional property is required: namely that the space-like hyper-surfaces admit an involution [76] allowing the definition

3) The super-hamiltonian constraint is the generator of the gauge transformations connecting different admissible 3+1 splittings of space-time and *has nothing to do with the temporal evolution*.

4) As shown by DeWitt [64], and already stressed by us, the weakly vanishing ADM Dirac Hamiltonian has to be modified with a suitable surface term in order that functional derivatives, Poisson brackets and Hamilton equations be mathematically well- defined in such non-compact space-times. This fact, in conjunction with the points 1), 2), 3) above, entails that *there is an effective evolution in the mathematical time* which parametrizes the leaves of the foliation associated with any 3+1 splitting. Such evolution is ruled by the weak *ADM energy* [44,63], i.e. by a non-vanishing Hamiltonian which exists also in the reduced phase space. This is the *rest-frame instant form of metric gravity*. As seen in Section III, each gauge fixing creates a realization of Γ_4 and the *weak ADM energy* is a functional of only the DO of that gauge. Then, the DO themselves (as any other function of them) satisfy the Hamilton equations

$$\dot{r}^G_{\bar{a}} = \{r^G_{\bar{a}}, E_{\rm ADM}\}^*, \quad \dot{\pi}^G_{\bar{a}} = \{\pi^G_{\bar{a}}, E_{\rm ADM}\}^*, \tag{5.3}$$

where $E_{\rm ADM}$ is intended as the restriction of the weak *ADM* Energy to Γ_4 and where the $\{\cdot,\cdot\}^*$ are Dirac Brackets.

5) When matter is present in this family of space-times, switching off Newton's constant ($G \mapsto 0$) yields the description of matter in Minkowski space-time foliated with the space-like hyper-planes orthogonal to the total matter 4-momentum (Wigner hyper-planes intrinsically defined by matter isolated system). In this way one gets the rest-frame instant form of dynamics reachable from parametrized Minkowski theories. Incidentally, this is the first example of consistent deparametrization of general relativity in which the ADM Poincaré group tends to the Poincaré group of the isolated matter system.

We can conclude that in these space-times there is neither a frozen reduced phase space nor a Wheeler-DeWitt interpretation based on some local concept of time like in compact space-times. Therefore, *our gauge-invariant approach to general relativity is perfectly adequate to accommodate objective temporal change.*

A further objection raised by Belot and Earmann [32] is that the *non-locality* of the gauge-invariant DO for the gravitational field is "philosophically unappealing". It may be, but, first of all, we can easily retort that this property is a unavoidable consequence[42] of establishing a well-posed Cauchy problem for any system of non-hyperbolic under-determined partial

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of a generalized Fourier transform with its associated concepts of positive and negative energy. This disproves the claimed impossibility of defining particles in curved space-times [77].

[42]Within the Hamiltonian approach; see for instance Ref. [65] for the simpler case of Yang-mills equations

differential equations, like Einstein's equations.[43] Second, just to the contrary, we believe that the specific property of DO for a pure gravitational field of being *non local in terms of manifold's mathematical points* is indeed philosophically appealing and displays the real physical content of Leibniz equivalence (see more in the Concluding Survey).

[43]This feature has a Machian flavor, although in a non-Machian context: with or without matter, the whole 3-space is involved in the definition of the observables. Furthermore, these space-times allow the separation [44] of the 4-center of mass of the universe (*decoupled point particle clock*) reminding the Machian statement that only relative motions are dynamically relevant. See Ref. [78] for a thorough discussion of Mach's influence on Einstein and for comments on the ontology of space-time (following from the hole argument) consistent with our interpretation.

VI. ON THE PHYSICAL INTERPRETATION OF DIRAC OBSERVABLES AND GAUGE VARIABLES: TIDAL-LIKE AND INERTIAL-LIKE EFFECTS.

Having settled the problem of the physical individuation of space-time points and that of the observables, let us discuss with a greater detail the physical meaning of the Hamiltonian gauge variables and DO.

As shown in Section III, the 20 off-shell canonical variables of the ADM Hamiltonian description are naturally subdivided into *two sets* by the quasi-Shanmugadhasan transformation:

i) The first set contains *seven off-shell Abelian Hamiltonian gauge variables* whose conjugate momenta are seven Abelianized *first class constraints*. The eighth canonical pair comprises the variable in which the super-hamiltonian constraint has to be solved (the conformal factor of the 3-metric, $\phi = ({}^3g)^{1/12}$) and its conjugate momentum as the eighth gauge variable. Precisely, the gauge variables are:

$$\pi_\phi, \quad momentum\, conjugate\, to\, the\, conformal\, factor\, (primary\, gauge\, variable),$$

$$N = \sqrt{\frac{{}^4g}{{}^3g}} = \sqrt{{}^4g_{\tau\tau} - {}^3g^{rs}\, {}^4g_{\tau r}\, {}^4g_{\tau s}}, \quad lapse\, function\, (its\, secondary\, gauge\, variable),$$

$$\xi^r, \quad (non-local)\, parameters\, of\, Diff_P\, \Sigma_\tau\, (primary\, gauge\, variables),$$

$$N_r = -{}^4g_{\tau r}, \quad shift\, functions\, (their\, secondary\, gauge\, variables). \tag{6.1}$$

Note that a "primary" gauge variable has its arbitrariness described by a Dirac multiplier, while a "secondary" gauge variable inherits the arbitrariness of the Dirac multipliers through the Hamilton equations.

ii) The second set contains the *off-shell gauge invariant (non-local and in general non tensorial) DO*: $r_{\bar a}(\tau, \vec\sigma)$, $\pi_{\bar a}(\tau, \vec\sigma)$, $\bar a = 1, 2$. They satisfy hyperbolic Hamilton equations and are not BO in general.

Let us stress again that the above subdivision of canonical variables in two sets is a peculiar outcome of the quasi-Shanmugadhasan canonical transformation which has no simple counterpart within the *Lagrangian viewpoint* at the level of the Hilbert action and/or of Einstein's equations: at this level the only clear statement is whether or not the curvature vanishes. As anticipated in the Introduction this subdivision amounts to an extra piece of (non-local) information which should be added to the traditional wisdom of the *equivalence principle* asserting the local impossibility of distinguishing gravitational from inertial effects. Indeed, we shall presently see that it allows to distinguish and visualize which aspects of the local physical effects on test matter contain a *genuine gravitational* component and which aspects depend solely upon the choice of the (local) reference frame and/or coordinate system: these latter could then be called *inertial*, in analogy with the non-relativistic Newtonian situation. Recall that when a complete choice of gauge is made, the gauge variables become fixed uniquely by the gauge-fixing procedure to functions of DO in that gauge.

One should be careful in discussing this subject because the very definition of *inertial force* (and *gravitational* as well) seems rather unnatural in general relativity. We can take advantage, however, from the circumstance that Hamiltonian point of view leads naturally to a re-reading of geometrical features in terms of the traditional concept of *force*.

First of all, recall that we are still considering here the case of pure gravitational field without matter. We know from Section III that on-shell, in any chosen 4-coordinate system, the mathematical representation of any physical effects is given in terms of functionals of the non-local DO in the completely fixed Hamiltonian gauge that corresponds to the chosen 4-coordinates. It is then natural first of all to characterize as genuine gravitational effects those which are directly correlated to the *DO*, and crucial to stress that such purely gravitational effects *are absent in Newtonian gravity*, where there are no autonomous gravitational fields, i.e., fields not generated by matter sources. It seems therefore plausible to trace *inertial* (much better than *fictitious*, in the relativistic case) effects to a pure dependence on the *Hamiltonian gauge variables*[44]. Recall also that, at the non-relativistic level, Newtonian gravity is fully described by *action-at-a-distance forces* and, in absence of matter, there are no *tidal* forces among test particles. Tidal-like forces are *entirely* determined by the variation of the action-at-a-distance force created by the Newton potential of a massive body on the test particles. In vacuum general relativity instead the *geodesic deviation equation* shows that tidal forces, locally described by the Riemann tensor, act on test particles even in absence of any kind of matter.

We can also say, therefore, that the role of the Hamiltonian gauge variables, whose form change from one gauge fixing to another, is that of describing the *form* in which physical gravitational effects determined by the DO show themselves. Such *appearances* undergo *inertial* changes upon going (on-shell) from one coordinate system (or gauge fixing) to another and above all from one reference frame to another (see later on). Furthermore, while *purely inertial* effects could be manifest even in case the DO are null, genuine gravitational

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[44]By introducing dynamical matter the Hamiltonian procedure leads to distinguish among *action-at-a-distance, gravitational, and inertial* effects, with consequent relevant implications upon concepts like gravitational passive and active masses and, more generally, upon the problem of the origin of inertia. See Ref. [79] for other attempts of separating inertial from tidal effects in the equations of motion in configuration space for test particles, in a framework in which asymptotic inertial observers are refuted. In this reference one finds also the following version (named Mach 11) of the Mach principle: "The so-called *inertial effects*, occurring in a non-inertial frame, are gravitational effects caused by the distribution and motion of the distant matter in the universe, relative to the frame". Thus *inertial* means here non-tidal + true gravitational fields generated by cosmic matter. In the above reference it is also suggested that superfluid Helium II may be an alternative to fixed stars as a standard of non rotation. Of course all these interpretations are questionable. On the other hand, the Hamiltonian framework offers the tools for making such a distinction while distant matter effects are hidden in the non-locality of DO and gauge variables. Since in a fixed gauge the gauge variables are functions of the Dirac obervables in that gauge, tidal effects are clearly mixed with inertial ones. For a recent critical discussion about the origin of inertia and its connection with inertial effects in accelerated and rotating frames see Ref. [80].

effects are always necessarily dressed by inertial-like appearances. Thus, the situation is only vaguely analogous to the phenomenology of non-relativistic inertial forces. These latter describe *purely apparent* (or really *fictitious*) mechanical effects which show up in accelerated Galilean reference frames [45] and can be eliminated by going to (global) inertial reference frames [46]. Besides the existence of autonomous gravitational degrees of freedom, it is therefore clear that the further deep difference concerning inertial-like forces in the general-relativistic case with respect to Newtonian gravity rests upon the purely *local nature* of general relativistic inertial frames of reference and/or coordinate systems.

For the sake of clarity, consider the non-relativistic Galilean framework in greater detail. If a *global non-inertial reference frame* has translational acceleration $\vec{w}(t)$ and angular velocity $\vec{\omega}(t)$ with respect to a given inertial frame, a particle with free motion ($\vec{a} = \ddot{\vec{x}} = 0$) in the inertial frame has the following acceleration as seen from the non-inertial frame

$$\vec{a}_{NI} = -\vec{w}(t) + \vec{x} \times \dot{\vec{\omega}}(t) + 2\dot{\vec{x}} \times \vec{\omega}(t) + \vec{\omega}(t) \times [\vec{x} \times \vec{\omega}(t)]. \tag{6.2}$$

After multiplication of this equation by the particle mass, the second term on the right hand side is the *Jacobi force*, the third is the *Coriolis force* and the fourth the *centrifugal force*.

We have given in Ref. [82] a description of non-relativistic gravity which is generally covariant under *arbitrary* passive Galilean coordinate transformations [$t' = T(t)$, $\vec{x}' = \vec{f}(t, \vec{x})$]. The analogue of Eq.(6.2) in this case contains more general *apparent* forces, which reduce to those appearing in Eq.(6.2) in particular rigid coordinate systems. The discussion given in Ref. [82] is a good introduction to the relativistic case, just because in general relativity there are no *global inertial reference frames*.

Two different approaches have been considered in the literature in the general relativistic case concerning the choice of reference frames, namely using either

i) a single accelerated time-like observer with an arbitrary associated tetrad,

or

ii) a congruence of accelerated time-like observers[47] with a conventionally chosen associ-

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[45]With arbitrary *global* translational and rotational 3-accelerations.

[46]See Ref. [81] for the determination of *quasi-inertial reference frames* in astronomy as those frames in which rotational and linear acceleration effects lie under the sensibility threshold of the measuring instruments.

[47]While in Newtonian physics an *absolute reference frame* is an imagined extension of a rigid body and a clock (with any coordinate systems attached), in general relativity [83] we must replace the rigid body either by a cloud of test particles in free fall (*geodesic congruence*) or by a test fluid (*non-geodesic congruence* for non-vanishing pressure). Therefore a *reference frame* is schematized as a future-pointing time-like congruence with all the possible associated 4-coordinate systems. This is

ated field of tetrads[48].

Usually, in both approaches the observers are *test observers*, which describe phenomena from their kinematical point of view without generating any dynamical effect on the system.

i) Consider first the case of a single *test observer* with his tetrad (see Ref. [88,89]).

After the choice of the associated local Minkowskian system of (Riemann-Gaussian) 4-coordinates, the line element becomes[49] $ds^2 = -\delta_{ij} dx^i dx^j + 2\epsilon_{ijk} x^j \frac{\omega^k}{c} dx^o dx^i + [1 + \frac{2\vec{a}\cdot\vec{x}}{c^2} (dx^o)^2]$. The *test observer* describes a nearby time-like geodesics $y^\mu(\lambda)$ (λ is the affine parameter or proper time) followed by a test particle in *free fall* in a given gravitational field by means of the following spatial equation: $\frac{d^2\vec{y}}{(dy^o)^2} = -\vec{a} - 2\vec{\omega} \times \frac{d\vec{y}}{dy^o} + \frac{2}{c^2}\Big(\vec{a} \cdot \frac{d\vec{y}}{dy^o}\Big) \frac{d\vec{y}}{dy^o}$. Thus, the relative acceleration of the particle with respect to the observer with this special system of coordinates[50] is composed by the observer 3-acceleration plus a relativistic correction and by a Coriolis acceleration[51]. Note that, from the Hamiltonian point of view, the constants $\vec{a}$ and $\vec{\omega}$ are constant functionals of the DO of the gravitational field in this particular gauge.

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called a *platform* in Ref. [84], where there is a classification of the possible types of platforms and the definition of the position vector of a neighboring observer in the local rest frame of a given observer of the platform. Then, the Fermi-Walker covariant derivative (applied to a vector in the rest frame it produces a new vector still in the rest frame [85]) is used to define the 3-velocity (and then the 3-acceleration) of a neighboring observer in the rest frame of the given observer, as the natural generalization of the Newtonian relative 3-velocity (and 3-acceleration). See Ref. [86] for a definition, based on these techniques, of the 3-acceleration of a test particle in the local rest frame of an observer crossing the particle geodesics, with the further introduction of the Lie and co-rotating Fermi-Walker derivatives.

[48]The time-like tetrad field is the 4-velocity field of the congruence. The conventional choice of the spatial triad is equivalent to a choice of a specific system of gyroscopes (see footnote 78 in Appendix B for the definition of a Fermi-Walker transported triad). See the local interpretation [86] of *inertial* forces as effects depending on the *choice of a congruence of time-like observers with their associated tetrad fields* as a reference standard for their description. Note that, in gravitational fields without matter, gravito-magnetic effects as described by ${}^4g_{\tau r}$ are purely inertial effects in our sense, since are determined by the *shift* gauge variables. While in metric gravity the tetrad fields are used only to rebuild the 4-metric, the complete theory taking into account all the properties of the tetrad fields is *tetrad gravity* [87].

[49]If the test observer is in free fall (*geodesic observer*) we have $\vec{a} = 0$. If the triad of the test observer is Fermi-Walker transported (standard of *non-rotation* of the gyroscope) we have $\vec{\omega} = 0$.

[50]It replaces the global non-inertial non-relativistic reference frame. With other coordinate systems, other terms would of course appear.

[51]This is caused by the rotation of the spatial triad carried by the observer relative to a Fermi-Walker transported triad. The vanishing of the Coriolis term justifies the statement that for an

As said above, different Hamiltonian gauge fixings on-shell, corresponding to on-shell variations of the Hamiltonian gauge variables, give rise to different appearances of the physical effects as gauge-dependent functionals of the DO in that gauge of the type $\mathcal{F}_G(r_{\bar{a}}, \pi_{\bar{a}})$ (like $\vec{a}$ and $\vec{\omega}$ in the previous example).

In absence of matter we can consider the *zero curvature limit*, which is obtained by putting the DO to zero. In this way we get Minkowski space-time (a solution of Einstein's equations) equipped with those kinds of coordinates systems which are compatible with Einstein's theory [52]. In particular, the quantities $\mathcal{F}_G = lim_{r_{\bar{a}},\pi_{\bar{a}} \rightarrow 0}\, \mathcal{F}_G(r_{\bar{a}}, \pi_{\bar{a}})$ describe inertial effects in those 4-coordinate systems for Minkowski space-time which have a counterpart in Einstein general relativity. Note finally that special relativity, considered as an autonomous theory, admits much more general inertial effects associated with the 3+1 splittings of Minkowski space-time whose leaves are not 3-conformally flat.

In presence of matter Newtonian gravity is recovered with a double limit:

a) the limit in which DO are restricted to the solutions of the Hamilton equations (5.3), $r_{\bar{a}} \rightarrow f_{\bar{a}}(matter)$, $\pi_{\bar{a}} \rightarrow g_{\bar{a}}(matter)$;
b) the $c \rightarrow \infty$ limit, in which curvature effects, described by matter after the limit a), disappear.

This implies that the functionals $\mathcal{F}_G(r_{\bar{a}}, \pi_{\bar{a}})$ must be restricted to the limit $\mathcal{F}_{Newton} = lim_{c \rightarrow \infty}\; lim_{r_{\bar{a}} \rightarrow f_{\bar{a}}, \pi_{\bar{a}} \rightarrow g_{\bar{a}}} \left(\mathcal{F}_{G\,o} + \frac{1}{c}\mathcal{F}_{G\,1} + ... \right) = \mathcal{F}_{G\,o}|_{r_{\bar{a}}=f_{\bar{a}}, \pi_{\bar{a}}=g_{\bar{a}}}$. Then $\mathcal{F}_{Newton}$, which may be coordinate dependent, becomes the *Newtonian inertial force* in the corresponding general Galilean coordinate system.

ii) Consider then the more general case of a *congruence of accelerated time-like observers*. In this way it is possible to get a much more accurate and elaborate description of the relative 3-acceleration, as seen in his own local rest frame by each observer of the congruence which intersects the geodesic of a test particle in free fall (see Ref. [86]). The identification of various types of 3-forces depends upon:

a) the gravitational field (the form of the geodesics obviously depends on the metric tensor; usually the effects of the gravitational field are classified as gravito-electric and gravito-magnetic, even if this is strictly valid only in harmonic coordinates) b) the properties

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observer which is not in free fall ($\vec{a} \neq 0$) a local coordinate system produced by Fermi-Walker transport of the spatial triad of vectors is the best possible realization of a non-rotating system.

[52] As shown in Ref. [44] this implies the vanishing of the Cotton-York 3-conformal tensor, namely the condition that the allowed 3+1 splittings of Minkowski space-time compatible with Einstein's equations have the leaves *3-conformally flat* in absence of matter. This solution of Einstein's equations, has been named *void space-time* in Ref. [87]: Minkowski space-time in Cartesian 4-coordinates is just a gauge representative of it. Note that, even if Einstein always rejected this concept, a void space-time corresponds to the description of a special class of 4-coordinate systems for Minkowski space-time without matter.

(acceleration, vorticity, expansion, shear) of the congruence of observers, c) the choice of the time-parameter used to describe the particle 3-trajectory in the local observer rest frame.

There are, therefore, many possibilities for defining the relative 3-acceleration (see Ref. [86]) and its separation in various types of inertial-like accelerations (See Appendix B for a more complete discussion of the properties of the congruences of time-like observers).

Summarizing, once a local reference frame has been chosen, in every 4-coordinate system we can consider:

a) the *genuine tidal gravitational effects* which show up in the geodesic deviation equation: they are well defined gauge-dependent functionals of the DO associated to that gauge: DO could then be called *non-local tidal-like degrees of freedom*;

b) the fact that *geodesic curves* will have different geometrical descriptions corresponding to different gauges, although they will be again functionally dependent only on the DO in the relevant gauge;

c) the issue of the description of the *relative 3-acceleration of a free particle in free fall*, as given in the local rest frame of a generic observer of the congruence, which will contain various terms. Such terms are identifiable with the general relativistic extension of the various non-relativistic kinds of inertial accelerations and all will again depend on the DO in the chosen gauge, both directly and through the Hamiltonian gauge variables of that gauge.

Three general remarks:

First of all, the picture we have presented is not altered by the presence of matter. The only new phenomenon besides the above purely gravitational, *inertial and tidal* effects, is that from the solution of the super-hamiltonian and super-momentum constraints emerge *action-at-a-distance, Newtonian-like and gravito-magnetic*, effects among matter elements, as already noted in footnote 44.

Secondly, we would like to recall that Bergmann [2] made the following critique of general covariance: it would be desirable to restrict the group of coordinate transformations (spacetime diffeomorphisms) in such a way that it could contain an invariant sub-group describing the coordinate transformations that change the frame of reference of an outside observer; the remaining coordinate transformations would be like the gauge transformations of electromagnetism. Yet, this is just what we have done with the redefinition of the lapse and shift functions after separating out their asymptotic part (see footnote 18). In this way *preferred* inertial asymptotic coordinate systems are selected which can be identified as fixed stars.

Thirdly, it should be stressed that the *reference standards* of time and length correspond to units of *coordinate time and length* and not to proper times and proper lengths [39]: this is not in contradiction with general covariance, because an extended *laboratory*, in which one defines the reference standards, corresponds to a particular *completely fixed on-shell Hamiltonian gauge* plus a local congruence of time-like observers. For instance, in astronomy and in the theory of satellites, the unit of time is replaced by a unit of coordinate

length (*ephemerides time*). This leads to the necessity of taking into account the theory of measurement in general relativity. This will be done in the next Section.

Fourthly, let us remark that the definitions given in Section V for the *notion of observable* in general relativity are in correspondence with the following two different points of view, existing in the physical literature, that are clearly spelled out in Ref. [90] and related references, namely:

i) The *non-local point of view* of Dirac [45], according to which causality implies that only gauge-invariant quantities, i.e., DO, can be measured. As we have shown, this point of view is consistent with general covariance. For instance, ${}^4R(\tau, \vec{\sigma})$ is a scalar under diffeomorphisms, and therefore a BO, but it is not a DO - at least the kinematical level - and therefore, according to Dirac, not an observable quantity. Even if ${}^4R(\tau, \vec{\sigma}) \stackrel{\circ}{=} 0$ in absence of matter, the other curvature scalars do not vanish in force of Einstein's equations and, lacking known solutions without Killing vectors, it is not clear their connection with the DO. The 4-metric tensor ${}^4g_{\mu\nu}$ itself as well as the line element ds^2 are not DO so a completely fixed gauge is needed to get a definite functional form for them in terms of the DO in that gauge. This means that all standard procedures for defining measures of length and time [20,85,60] and the very definition of angle and distance properties of the material bodies forming the reference system, are gauge dependent. Then they are determined only after a complete gauge fixing and after the restriction to the solutions of Einstein's equations has been made[53]. Likewise, it is only after the gauge fixing that the procedure for measuring the Riemann tensor with $n \geq 5$ test particles described in Ref. [60] (see also Ref. [31]) becomes completely meaningful, just as it happens for the electro-magnetic vector potential in the radiation gauge.

Finally note that, after the introduction of matter, even the measuring apparatuses should be described by the gauge invariant matter DO associated with the given gauge.

ii) The *local point of view*, according to which the space-time manifold M^4 is a kind of postulated (often without any explicit statement) background manifold of physically determinated *events*, like it happens in special relativity with its absolute chrono-geometric structure. Space-time points are assumed physically distinguishable, because any measurement is performed in the frame of a given reference system interpreted as a physical laboratory. In this view the gauge freedom of generally covariant theories is reduced to mere passive coordinate transformations. See for instance Ref. [91] for a refusal of DO in general relativity based on the local point of view. However this point of view ignores completely the Hole Argument and must renounce to a deterministic evolution, so that it is ruled out definitely by our results.

In Ref. [90] the non-local point of view is accepted and there is a proposal for using some special kind of matter to define a *material reference system* (not to be confused with a coordinate system) to localize points in M^4, so to recover the local point of view in some

[53]Note that in textbooks these procedures are always defined without any reference to Einstein's equations.

approximate way [54] because in the analysis of classical experiments both approaches lead to the same conclusions. This approach relies therefore upon *matter* to solve the problem of the individuation of space-time points as point-events, at the expense of loosing determinism. The emphasis on the fundamental role of matter for the individuation issue points is present also in Refs. [92,93,33,35], where *material clocks* and *reference fluids* are exploited as test matter.

As we have shown, the problem of the individuation can be solved *before and without* the introduction of matter. Matter only contributes to a deformed individuation and, obviously, is fundamental in trying to establish a general-relativistic theory of measurement.

[54] The main approximations are: 1) to neglect, in Einstein equations, the energy-momentum tensor of the matter forming the material reference system (it's similar to what happens for test particles); 2) to neglect, in the system of dynamical equations, the entire set of equations determining the motion of the matter of the reference system (this introduces some *indeterminism* in the evolution of the entire system).

VII. IMPLEMENTING THE PHYSICAL INDIVIDUATION OF POINT-EVENTS WITH WELL-DEFINED EMPIRICAL PROCEDURES: A REALIZATION OF THE AXIOMATIC STRUCTURE OF EHLERS, PIRANI AND SCHILD.

The problem of the individuation of space-time points as point-events cannot be methodologically separated from the problem of defining a theory of measurement consistent with general covariance. This means that we should not employ the absolute chrono-geometric structures of special relativity, like it happens in all the formulations on a given background (gravitational waves as a spin two field over Minkowski space-time, string theory,...). Moreover matter (either test or dynamical) is now an essential ingredient for defining the experimental setup.

At present we do not have such a theory, but only preliminary attempts and an empirical metrology [39], in which the standard unit of time is a *coordinate time* and not a proper time. As already said, a laboratory network with its standards corresponds to a description givn in a completely fixed Hamiltonian gauge viz., being on-shell, in a uniquely determined 4-coordinate system. We shall take into account the following pieces of knowledge.

A) Ehlers, Pirani and Schild [40] developed an axiomatic framework for the foundations of general relativity and measurements (reviewed in Appendix C for completeness). These authors exploit the notions of *test objects* as *idealizations* to the effect of approximating the *conformal, projective, affine* and *metric* structures of Lorentzian manifolds; such structures are then used to define ideal geodesic clocks [88]. The axiomatic structure refers to basic objects such as *test light rays* and *freely falling test particles*. The first ones are used in principle to reveal the *conformal* structure of space-time, the second ones the *projective* structure. Under an axiom of compatibility which is well corroborated by experiment (see Ref. [94]), it can be shown that these two independent classes of observations determine completely the structure of space-time. Let us remark that one should extend this axiomatic theory to tetrad gravity (space-times with frames) in order to include objects like *test gyroscopes* needed to detect gravito-magnetic effects.[55].

B) De Witt [92] introduced a procedure for measuring the gravitational field based on a reference fluid (a stiff elastic medium) equipped with material clocks. This phenomenological test- fluid is then exploited to bring in Bergmann-Komar invariant pseudo-coordinates $\zeta^{\bar{A}}$, $\bar{A} = 1,..,4$, as a method for coordinatizing the space-time where to do measurements and also for grounding space-time geometry operationally, at least in the weak field regime. De Witt essentially proposes to simulate a mesh of local clocks and rods. Even if De Witt considers the measurement of a weak quantum gravitational field smeared over such a region, his procedure could even be adopted classically. In this perspective, our approach furnishes the ingredients of the Hamiltonian description of the gravitational field, which were lacking at the time De Witt developed his preferred covariant approach.

C) Antennas and interferometers are the tools used to detect gravitational waves on the earth. The mechanical prototype of these measurements are test springs with end masses

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[55] Stachel [16], stresses the dynamical (not axiomatic) aspect of the general relativistic space-times structures associated to the behavior of ideal measuring rods (*geometry*) and clocks (*chronometry*) and free test particles (*inertial structures*)

feeling the gravitational field as the tidal effect described by the geodesic deviation equation [88,95]. Usually, however, one works on the Minkowski background in the limit of weak field and non-relativistic velocities. See Ref. [96] for the extension of this method to a regime of weak field but with relativistic velocities in the framework of a background-independent Hamiltonian linearization of tetrad gravity.

Lacking solutions to Einstein's equations with matter corresponding to simple systems to be used as idealizations for a measuring apparatuses described by matter DO (hopefully also BO), a generally covariant theory of measurement as yet does not exist. We hope, however, that some of the clarifications achieved in this paper of the existing ambiguities about observables will help in developing such a theory.

In the meanwhile we want to sketch in this last Section a scheme for implementing - at least in principle - the physical individuation of points as an experimental setup and protocol for positioning and orientation. Our construction should be viewed in parallel to the axiomatic treatment of Ehlers, Pirani and Schild. We could reproduce the logical scheme of this axiomatic approach in the following way.

a) A *radar-gauge* system of coordinates can be defined in a finite four-dimensional volume by means of a network of artificial satellites similar to the Global Position System [97]. Let us consider a family of spacecrafts, whose navigation is controlled from the Earth by means of the standard GPS. Note that the GPS receivers are able to determine their actual position and velocity because the GPS system is based on the advanced knowledge of the gravitational field of the Earth and of the satellites' trajectories, which in turn allows the *coordinate* synchronization of the satellite clocks . During the navigation the spacecrafts are test objects. Once the spacecrafts have arrived in regions with non weak fields, like near the Sun or Jupiter, they become the (non test) elements of an experimental setup and protocol for the determination of a local 4-coordinate system and of the associated 4-metric.

Each satellite, endowed with an atomic clock and a system of gyroscopes, may be thought as a time-like observer (the satellite world-line) with a tetrad (the time-like vector is the satellite 4-velocity and the spatial triad is built with gyroscopes) and one of them is chosen as the origin of the radar-4-coordinates we want to define. This means that the natural framework should be tetrad gravity instead of metric gravity.

Since the geometry of space-time and the motion of the satellites are not known in advance in our case, we must think of the receivers as obtaining four, so to speak, *conventional* coordinates by operating a full-ranging protocol involving bi-directional communication to four *super-GPS* that broadcast the time of their standard unsynchronized clocks (see the discussion given in Ref. [5] and Refs. [98] for other proposals in the same perspective). This first step parallels the axiomatic construction of the *conformal structure* of space-time.

b) Then, choose Einstein's simultaneity convention to synchronize the atomic clocks by means of radar signals [99], with respect to the satellite chosen as origin: this allows establishing a *radar-gauge system of 4-coordinates* lacking any direct metrical content

$$\sigma^A_{(R)} = (\tau_{(R)}; \sigma^r_{(R)}), \tag{7.1}$$

in a finite region ($\tau_{(R)} = const$ defines the radar simultaneity surfaces). Then the navigation system provides determination of the 4-velocities (time-like tetrads) of the satellites and the ${}^4g_{(R)\tau\tau}$ component of the 4-metric in these coordinates.

Note that by replacing *test radar signals* (conformal structure) with *test particles* (projective structure) in the measurements, we would define a different 4-coordinate system.

In the framework of metric gravity suitable spacecrafts make repeated measurements of the motion of *four* test particles. In this way we test also the *projective structure* in a region of space-time with a vacuum gravitational field). By the motion of gyroscopes we measure the *shift* components ${}^4g_{(R)\tau r}$ of the 4-metric) and end up (in principle) with the determination of all the *components of the four-metric* with respect to the *radar-gauge* coordinate system:

$$ {}^4g_{(R)AB}(\tau_{(R)}, \sigma^r_{(R)}). \tag{7.2}$$

The tetrad gravity alternative, employing test gyroscopes and light signals (i.e. only the *conformal structure*), is the following. By means of exchanges (two-ways signals) of *polarized* light it should be possible to determine how the spatial triads of the satellites are rotated with respect to the triad of the satellite chosen as origin (see also Ref. [100]). Once we have the tetrads ${}^4E^A_{(r)(\alpha)}(\tau_{(R)}, \vec{\sigma}_{(R)})$ in radar coordinates, we can build from them the inverse 4-metric ${}^4g^{AB}_{(R)}(\tau_{(R)}, \vec{\sigma}_{(R)}) = {}^4E^A_{(r)(\alpha)}(\tau_{(R)}, \vec{\sigma}_{(R)})\, {}^4\eta^{(\alpha)(\beta)}\, {}^4E^B_{(r)(\beta)}(\tau_{(R)}, \vec{\sigma}_{(R)})$ in radar coordinates.

c) By measuring the spatial and temporal variation of ${}^4g_{(R)AB}$, the components of the Weyl tensor and the Weyl eigenvalues can in principle be determined.

d) Points a), b) and c) furnishes *operationally* a slicing of space-time into surfaces $\tau_{(R)} = const$, a system of coordinates $\sigma^r_{(R)}$ on the surfaces, as well as a determination of the components of the metric ${}^4g_{(R)AB}$. The components of the Weyl tensor (= Riemann in void) and the local value of the Weyl eigenvalues, with respect to the radar-gauge coordinates $(\tau_{(R)}, \sigma^r_{(R)})$ are also thereby determined. It is then a matter of computation:

i) to check whether Einstein's equation in radar-gauge coordinates are satisfied;

ii) to get a numerical determination of the intrinsic coordinate functions $\bar{\sigma}^{\bar{A}}_R$ defining the radar gauge by the gauge fixings $\sigma^A_R - \bar{\sigma}^{\bar{A}}_R(\sigma_R) \approx 0$. Since we know the eigenvalues of the Weyl tensor in the radar gauge, it is possible to solve in principle for the functions $F^{\bar{A}}$ that reproduce the *radar-gauge* coordinates as *radar-gauge intrinsic* coordinates

$$\sigma^A_{(R)} = F^{\bar{A}}[\tilde{\Lambda}^{(k)}_W[{}^3g(\tau, \vec{\sigma}), {}^3\Pi(\tau, \vec{\sigma})]], \tag{7.3}$$

consistently with the *gauge-fixing* that enforces just this particular system of coordinates:

$$\bar{\chi}^A(\tau, \vec{\sigma}) \stackrel{def}{=} \sigma^A - \bar{\sigma}^{\bar{A}}(\tau, \vec{\sigma}) = \sigma^A - F^{\bar{A}}\Big[\tilde{\Lambda}^{(k)}_W[{}^3g(\tau, \vec{\sigma}), {}^3\Pi(\tau, \vec{\sigma})]\Big] \approx 0. \tag{7.4}$$

Finally, the *intrinsic coordinates* are reconstructed as functions of the *DO* of the radar gauge, at each point-event of space-time, as the identity

$$\sigma^A \equiv \bar{\sigma}^{\bar{A}} = \tilde{F}^{\bar{A}}_G[r^{(R)}_{\bar{a}}(\tau, \vec{\sigma}), \pi^{(R)}_{\bar{a}}(\tau, \vec{\sigma})], \tag{7.5}$$

This procedure of principle would close the *coordinate circuit* of general relativity, linking individuation to operational procedures [5].

VIII. CONCLUDING SURVEY

The main results of our investigation are:

1) A definite procedure for the *physical individuation* of the mathematical points of the would-be space-time manifold M^4 into physical *point-events*, through a gauge-fixing identifying the mathematical 4-coordinates with the *intrinsic pseudo-coordinates* of Komar and Bergmann (defined as suitable functionals of the Weyl scalars). This has led to the conclusion that each of the point-events of space-time is endowed with its own physical individuation as the value, as it were, at that point, of the four canonical coordinates or Dirac observables (just four!), which describe the dynamical degrees of freedom of the gravitational field. Since such degrees of freedom are non-local functionals of the metric and curvature, they are unresolveably entangled with the whole texture of the metric manifold in a way that is strongly both gauge-dependent and highly non-local with respect to the background mathematical coordinatization. We can also say, on the other hand, that *any coordinatization* of the manifold can be seen as embodying the physical individuation of points, because it can be implemented as the Komar-Bergmann *intrinsic pseudo-coordinates*, after a suitable choice of the functions of the Weyl scalars and of the gauge-fixing. We claim that our results bring the Synge-Bergmann-Komar-Stachel program of the physical individuation of space-time points to its natural end.

2) A clarification of Bergmann's multiple ambiguous definition of *observable* in general relativity. This has led to formulate our *main conjecture* concerning the unification of Bergmann's and Dirac's concepts of observable, as well as to restate the issue of change and, in particular and independently, of temporal change, within the Hamiltonian approach to Einstein equations. When concretely carried out, this program would provide even *explicitly* evidence for the *invariant objectivity* of point-events. Furthermore, the existence of simultaneous *Bergmann-Dirac observables* and *PDIQ gauge variables* would lead to a description of tidal-like and inertial-like effects in a coordinate independent way, while the *Dirac-Bergmann* observables only would remain as the only quantities subjected to a causal evolution. If the conjecture about the existence of simultaneous DO-BO observables is sound, it would open the possibility of a new type of coordinate-independent canonical quantization of the gravitational field. Only the DO should be quantized in this approach, while the gauge variables, i.e., the *appearances* of inertial effects, should be treated as c-number fields (a prototype of this quantization procedure is under investigation [101] in the case of special relativistic and non-relativistic quantum mechanics in non-inertial frames). This would permit to preserve causality (the space-like character of the simultaneity Cauchy 3-surfaces), the property of having only the 3-metric quantized (with implications similar to loop quantum gravity for the quantization of spatial quantities), and to avoid any talk of *quantization of the 4-geometry* (see more below), a talk we believe to be deeply misleading (in this connection see Ref. [102])

The key technical tool which made such results possible has been the re-visitation of a nearly forgotten paper by Bergmann and Komar concerning the most general *Coordinate Group Symmetries of General Relativity* [14]. We have made explicit that this paper establishes the general framework for characterizing the correspondence between the *active*

diffeomorphisms operating in the configurational manifold M^4, on the one hand, and the *on-shell gauge transformations* of the ADM canonical approach to general relativity, on the other. Understanding such correspondence is fundamental for fully disclosing the connection of the Hole phenomenology, at the Lagrangian level, with the correct formulation of the initial value problem of the theory and its gauge invariance, at the Hamiltonian level. The upshot is the discovery that, as concerns both the Hole Argument and the issue of *predictability*, not all active diffeomorphisms of M^4 can play an effective role, because not all of them satisfy the bounds imposed by a correct mathematical setting of the initial value problem at the Lagrangian level.

3) An illustration of the physical role played by the Dirac observables and the gauge variables to the effect of characterizing intrinsic *tidal effects* and *inertial effects*, respectively.

4) An outline of the implementation (in principle) of the physical individuation of point-events as an experimental setup and protocol for positioning and orientation, which closes, as it were, the empirical *coordinative circuit* of general relativity.

We want to conclude our discussion with some general conceptual remarks about the foundation of general relativity and some venturesome suggestions concerning quantum gravity. First of all, our program is substantially grounded upon the *gauge nature* of general relativity. Such property of the theory, however, is far from being a simple matter and we believe that it is not usually spelled out in a sufficiently explicit and clear fashion. The crucial point is that general relativity *is not* a standard gauge theory like, e.g., electromagnetism or Yang-Mills theories in some relevant respects. Let us recall the very general definition of gauge theory given by Henneaux and Teitelboim [48]

> These are theories in which the physical system being dealt with is described by more variables than there are physically independent degrees of freedom. The physically meaningful degrees of freedom then re-emerge as being those invariant under a transformation connecting the variables (gauge trasformation). Thus, one introduces extra variables to make the description more transparent, and brings in at the same time a gauge symmetry to extract the physically relevant content.

The relevant fact is that, while from the point of view of the constrained Hamiltonian formalism general relativity is a gauge theory like any other, it is radically different from the physical point of view. For, in addition to creating the distinction between what is observable[56] and what is not, the gauge freedom of general relativity is unavoidably entangled with the definition–constitution of the very *stage*, space–time, where the *play* of physics is enacted. In other words, the gauge mechanism has the double role of making the dynamics unique (as in all gauge theories), and of fixing the spatio-temporal reference background. It is only after a complete gauge-fixing (i.e. after the individuation of a *well defined* physical laboratory network) is specified, and after going on shell, that even the mathematical

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[56] In the Dirac or Bergmann sense.

manifold M^4 gets a *physical individuation* and becomes the spatio-temporal carrier of well defined physical generalized *tidal-like* (DO) and *inertial-like* (gauge variables) effects.

In gauge theories such as electromagnetism (or even Yang-Mills), we can rely from the beginning on empirically validated, gauge-invariant dynamical equations for the *local* fields. This is not the case for general relativity: in order to get dynamical equations for the basic field in a *local* form, we must pay the price of Einstein's general covariance which, by ruling out any background structure at the outset, weakens the objectivity that the spatiotemporal description could have had *a priori*.

With reference to the definition of Henneaux and Teitelboim, we could say, therefore, that the introduction of extra variables does make the mathematical description of general relativity more transparent, but it also makes its physical interpretation more obscure and intriguing, at least at first sight.

The isolation of the superfluous structure hidden behind Leibniz equivalence, which surface in the physical individuation of point-events, renders even more glaring the ontological diversity and prominence of the gravitational field with respect to all other fields, as well as the difficulty of reconciling the deep nature of the gravitational field with the standard wisdom of theories based on background space-time. Any procedure of linearizing these latter unavoidably leads to looking at gravity as to a spin-2 theory in which the graviton stands on the same ontological level of other quanta: in the standard approach, photons, gluons and gravitons all live on the stage on equal footing. From the point of view we gained in this paper, however, *non-linear gravitons* do in fact constitute the stage for the causal play of photons, gluons as well as of other matter actors like electrons and quarks. More precisely, if our main conjecture is sound, the *non-linear graviton* would be represented by a pair of *scalar fields*. Finally, concerning the "background" philosophy, it's worth recalling that Steven Weinstein [102] has aptly remarked that what ultimately makes unattractive viewing the gravitational field as a simple distribution of properties (the field *strengths*) in *flat space-time*, on the same footing of all other fields, is the fact that, because of the universal nature of gravitation, the distinctive properties of this background space-time would be *completely unobservable*.

It should be clear by now that the Hole Argument has little to do with an alleged *indeterminism* of general relativity as a dynamical theory. For, in our analysis of the initial-value problem within the Hamiltonian framework, we have shown that *on shell* a complete gauge-fixing (which could in theory concern the whole space-time) is equivalent to the choice of an atlas of coordinate charts on the space-time manifold, and in particular *within the Hole*. At the same time, we have shown that a peculiar subset of the active diffeomorphisms of the manifold can be interpreted as passive Hamiltonian gauge transformations. Because the gauge must be fixed *before* the initial-value problem can be solved to obtain a solution (outside and inside the Hole), it makes little sense to apply active diffeomorphisms to an already generated solution to obtain an allegedly "different" space-time. Conversely, it is possible to generate these "different" solutions by appropriate choices of the initial gauge fixing. Of course, the price to be payed for the physical individuation of the *stage* is the breaking of general covariance.

We can, therefore, say that general covariance represents a horizon of *a priori* possibilities for the physical constitution of the space-time, possibilities that must be actualized within any given solution of the dynamical equations. Of course, what we call here *physical*

constitution embodies at the same time the *chrono-geometrical*, the *gravitational*, and the *causal* properties of the space-time stage.

We believe in conclusion that these results cast some light over the *intrinsic structure* of the general relativistic space-time that had disappeared within Leibniz equivalence and that was the object of Michael Friedman's non-trivial question[57].

In 1972, Bergmann and Komar wrote [14]:

> [...] in general relativity the identity of a world point is not preserved under the theory's widest invariance group. This assertion forms the basis for the conjecture that some physical theory of the future may teach us how to dispense with world points as the ultimate constituents of space-time altogether.

Indeed, would it be possible to build a fundamental theory that is grounded in the reduced phase space parametrized by the Dirac observables? This would be an abstract and highly nonlocal theory of gravitation that would admit an infinity of gauge-related, spatio-temporally local realizations. From the mathematical point of view, this theory would be just an especially perspicuous instantiation of the relation between canonical structure and locality that pervades contemporary theoretical physics nearly everywhere.

On the other hand, beyond the mathematical transparency and the latitude of choices guaranteed by general covariance, we need to remember that *local* spatio-temporal realizations of the abstract theory would still be needed for implementation of measurements in practice; conversely, any real-world experimental setting entails the choice of a definite *local realization*, with a corresponding gauge fixing that breaks general covariance.

Can this basic freedom in the choice of the *local realizations* be equated with a "taking away from space and time the last remnant of physical objectivity," as Einstein suggested? We believe that if we strip the physical situation from Einstein's "spatial obsession" about *realism as locality (and separability)*, a significant kind of spatio-temporal objectivity survives. It is true that the *functional dependence* of the Dirac observables upon the spatio-temporal coordinates depends on the particular choice of the latter (or equivalently, of the gauge); yet, there is no a-priori *physical* individuation of the points independently of the metric field, so we cannot say that the physical-individuation procedures corresponding to different gauges individuate physical point-events that are *really* different. Given the conventional nature of the primary standard mathematical individuation of manifold points through n-tuples of

[57]It is rather curious to recall here the following passage of Leibniz: "Space being uniform, there can be neither any external nor internal reason, by which to distinguish its parts, and to make any choice between them. For, any external reason to discern between them, can only be grounded upon some internal one. Otherwise we should discern what is indiscernible, or choose without discerning" [103]. Clearly, if the parts of space were real, the Principle of Sufficient Reason would be violated. Therefore, for Leibniz, space is not real. The upshot, however, is that space (space-time) in general relativity is not *uniform* and this is just the reason why - in our sense - it is *real*. Thus, Leibniz equivalence called upon for general relativity happens to hide the very nature of space-time, instead of disclosing it.

real numbers, we could say instead that the *real point-events* are constituted by the non-local values of gravitational degrees of freedom, while the underlying point structure of the mathematical manifold may be changed at will.

Taking into account our results on the whole, we want to spend a few additional words about the consequences that the acquired knowledge entail for the concept of space-time of general relativity, as seen in the wider context of the traditional debate on the absolutist/relationist dichotomy.

First of all, let us recall that, in remarkable diversity with respect to the traditional historical presentation of Newton's absolutism due to the influence of Leibniz, Newton himself had in fact a deeper understanding about the reality of space and time with respect to what has been traditionally ascribed to his absolutism. In a well-known passage of *De Gravitatione* [104], Newton expounds what could be defined an original *structuralist view* of space-time (see also Ref. [105]. He writes:

> Perhaps now it is maybe expected that I should define extension as substance or accident or else nothing at all. But by no means, for it has its own manner of existence which fits neither substance nor accidents [...] Moreover the immobility of space will be best exemplified by duration. For just as the parts of duration derive their individuality from their order, so that (for example) if yesterday could change places with today and become the latter of the two, it would lose its individuality and would no longer be yesterday, but today; so the parts of space derive their character from their positions, so that if any two could change their positions, they would change their character at the same time and each would be converted numerically into the other *qua* individuals. The parts of duration and space are only understood to be the same as they really are because of their mutual order and positions (*propter solum ordinem et positiones inter se*); nor do they have any other *principle of individuation* besides this order and position which consequently cannot be altered.

We have just disclosed the fact that the points of general-relativistic space-times, quite unlike the points of the homogeneous Newtonian space, are endowed with a remarkably rich *non-point-like* texture furnished by the metric field. Therefore, the general-relativistic metric field itself or, better, its independent degrees of freedom, have the capacity of characterizing the "mutual order and positions" of points *dynamically*, and in fact much more than this.

Two important remarks are in order. First: such dynamical degrees of freedom are non-local functionals of the 3-metric and curvature [58] so that they are unresolveably entangled with the whole texture of the metric manifold in a way that is strongly both gauge-dependent

[58]Admittedly, at least at the classical level, we don't know of any detailed analysis of the relationship between the notion of *non-local observable* (the predictable degrees of freedom of a gauge system), on one hand, and the notion of a quantity which has to be operationally *measurable* by means of *local apparatuses*, on the other. Note that this is true even for the simple case of the electro-magnetic field where the Dirac observables are defined by the *transverse* vector potential and the *transverse* electric field. Knowledge of such fields at a definite mathematical time involves data on the whole Cauchy surface at that time. Even more complex is the situation in the case of Yang-Mills theories [65]

and highly non-local. Still, once they are calculated, they appear as *local fields* in terms of the background *mathematical* coordinatization, a fact that makes the identity Eq.(4.8) possible. Contrary to the opinion of Belot and Earman [32], from our point of view the peculiar non-locality of the Dirac observables is therefore philosophically appealing and constitutes an asset rather than a liability of our approach; and shows, in a sense, a Machian flavor within a non-Machian environment.

Second, consider the ADM approach to Einstein's equations for the gravitational field *cum* matter. In this case we have Dirac observables both for the gravitational field and for the matter fields, but *the former are modified in their functional form* with respect to the vacuum case by the *presence of matter*. Since the gravitational Dirac observables will still provide the individuating fields for point-events according to the conceptual procedure presented in this paper, *matter will come to influence the very physical individuation of points*.

In conclusion, we agree with Earman and Norton that the Hole phenomenology constitutes a decisive argument against strict manifold substantivalism. However, the isolation of the intrinsic structure hidden within Leibniz equivalence does not support the standard relationist view either. With reference to the third criterion (R_3) stated by Earman for relationism (see Ref. [13], p.14): "No irreducible, monadic, spatiotemporal properties, like 'is located at space-time point p' appears in a correct analysis of the spatiotemporal idiom", we observe that if by 'space-time point' we mean our physically individuated point-events instead of the naked manifold's point, then - because of the autonomous existence of the intrinsic degrees of freedom of the gravitational field (an essential ingredient of general relativity) - the quoted spatiotemporal property should be admitted in our spatiotemporal idiom.

In conclusion, what emerges from our analysis is rather a kind of *new structuralist* conception of space-time. Such new structuralism is not only richer than that of Newton, as it could be expected because of the dynamical structure of Einstein space-time, but richer in an even deeper sense. For this *new structuralist* conception turns out to include elements common to the tradition of both *absolutism* (space has an autonomous existence independently of other bodies or matter fields) *and relationism* (the physical meaning of space depends upon the relations between bodies or, in modern language, the specific reality of space depends (also) upon the (matter) fields it contains).

Let us close this survey with some hints that our results tend to suggest for the quantum gravity programme. As well-known this programme is documented nowadays by two inequivalent quantization methods: i) the perturbative background-dependent *string* formulation, on a Fock space containing elementary particles; ii) the non-perturbative background-independent *loop* quantum gravity approach, based on the non-Fock *polimer* Hilbert space. In this connection, see Ref. [105] for an attempt to define a *coarse-grained structure* as a bridge between standard *coherent states* in Fock space and some *shadow states* of the discrete quantum geometry associated to a *polimer* Hilbert space. As well-known, this approach still fails to accomodate elementary particles.

Now, the individuation procedure we have proposed transfers, as it were, the non-commutative Poisson-Dirac structure of the Dirac observables onto the individuated point-events even if, of course, the coordinates on the l.h.s. of the identity Eq.(4.8) are c-numbers

quantities. Of course, no direct physical meaning can be attributed to this circumstance at the classical level. One could guess, however, that such feature might deserve some attention in view of quantization, for instance by maintaining that the identity (Eq.(4.8)) could still play some role at the quantum level. We will assume here that the *main conjecture* is verified so that all the quantities we consider are manifesttly covariant.

Let us first lay down some qualitative premises concerning the status of Minkowski space-time in relativistic quantum field theory (call it *micro space-time*, see Ref. [106]). Such status is quite peculiar. From the chrono-geometric point of view, the micro space-time is a universal, classical, non-dynamical space-time, just Minkowski's space-time of the special theory of relativity, utilized without any scale limitation from below. However, it is introduced into the theory through the group-theoretical requirement of relativistic invariance of the *statistical* results of measurements with respect to the choice of *macroscopic reference frames*. The micro space-time is therefore *anchored* to the macroscopic medium-seized objects that asymptotically define the experimental conditions in the laboratory. It is, in fact, in this asymptotic sense that a physical meaning is attributed to the classical spatiotemporal *coordinates* upon which the quantum fields' operators depend as *parameters*. Thus, the spatiotemporal properties of the *micro Minkowski manifold*, including its basic causal structure, are, as it were, projected on it *from outside*[59].

In classical field theories space-time points play the role of individuals and we have seen how the latter can be individuated dynamically. No such possibility, however, is consistently left open in a non-metaphoric way in relativistic quantum field theory. From this point of view, Minkowski's *micro space-time* is in a worse position than general relativistic space-time: it lacks the existence of Riemannian *intrinsic pseudo-coordinates*, as well as all of the non-dynamical (better, operational and pragmatical) additional elements that are being used for the individuation of its points, like rigid rods and clocks in rigid and un-accelerated motion, or various combinations of *genidentical* world-lines of free test particles, light rays, clocks, and other devices. Summarizing, the role of *Minkowski's micro space-time* seems to be essentially that of an instrumental *external translator* of the symbolic structure of quantum theory into the *causal* language of the macroscopic irreversible traces that constitute

[59]In view of these circumstances, it could appear *prima facie* miraculous that the quantum-relativistic micro-causality conditions and the standard gauge theories whose symmetries depend on the points of the *micro space-time*, have had such an extraordinary empirical success. But this should appear less miraculous if one notes that the most consistent quantum field theories seem to be the so-called asymptotically-free non-Abelian gauge theories. If we interpret *physical observability* in this context as possibility of probing micro-structure by means of interactions, we should conclude that such gauge theories are, in fact, consistent to the extent that they are insensitive to short-scale space-time behavior, a circumstance that does not hold in the case of gravitational interaction. In this sense, each point of the *micro-space-time* could be coherently understood, so to speak, as a representative *compendium* of something spatiotemporal but not continuously extended in the traditional sense of the manifold M^4. Also, one should not forget that the Minkowski structure of the *micro-space-time* has been probed down to the scale of 10^{-18} m., yet only from the point of view of *scattering* experiments, involving a limited number of *real* particles.

the experimental findings within *macro space-time*. The conceptual status of this *external translator* fits then very well with that of epistemic precondition for the formulation of relativistic quantum field theory in the sense of Bohr, independently of one's attitude towards the interpretation of quantum theory of measurement. Thus, barring macroscopic Schrödinger Cat states of the would-be quantum space-time, any conceivable formulation of a quantum theory of gravity would have to respect, at the *operational* level, the *epistemic priority* of a classical spatiotemporal continuum. Talking about the quantum structure of space-time needs overcoming a serious conceptual difficulty concerning the localization of the gravitational field: indeed, what does it even mean to talk about the *values* of the gravitational field *at a point*, to the effect of points individuation, if the field itself is subject to quantum fluctuations ? One needs in principle some sort of reference structure in order to give physical operational meaning to the spatiotemporal language, one way or the other. This instrumental background, mathematically represented by a manifold structure, should play, more or less, the role of the Wittgenstein's staircase. It is likely, therefore, that in order to attribute some meaning to the individuality of points that lend themselves to the basic structure of standard quantum theory, one should split, as it were, the individuation of point-events from the true quantum properties, i.e., from the fluctuations of the gravitational field and the micro-causal structure. Now, it seems that our canonical analysis of the individuation issue, tends to prefigure a new approach to quantization having in view a Fock space formulation. Accordingly, unlike loop quantum gravity, this approach could even lead to a background-independent incorporation of the standard model of elementary particles (provided the Cauchy surfaces admit Fourier transforms). Two options present themselves for a quantization program respecting relativistic causality [60]:

1) The procedure for the individuation outlined in Section IV suggests to quantize the DO=BO of each Hamiltonian gauge, as well as all the matter DO, and to use the weak ADM energy of that gauge as Hamiltonian for the functional Schrödinger equation (of course there might be ordering problems). This quantization would yield as many Hilbert spaces as 4-coordinate systems, which would likely be grouped in unitary equivalence classes (we leave aside asking what could be the meaning of inequivalent classes, if any). In each Hilbert space the DO=BO quantum operators would be distribution-valued quantum fields on a *mathematical micro space-time* parametrized by the 4-coordinates τ, $\vec{\sigma}$ associated to the chosen gauge. Strictly speaking, due to the non-commutativity of the operators $\hat{\tilde{F}}^{\bar{A}}$ associated to the classical gauge-fixing (4.5) $\sigma^A - F^{\bar{A}} \approx 0$ defining that gauge, there would be *no space-time manifold of point-events* to be mathematically identified by one coordinate chart over the *micro-space-time*: only a *gauge-dependent non-commutative structure* which is likely to lack any underlying topological space structure. However, for each Hilbert space, a *coarse-grained* space-time of point-events might be associated to each solution of the functional Schrödinger equation, through the expectation values of the operators $\hat{\tilde{F}}^{\bar{A}}$:

$$\bar{\Sigma}^{\bar{A}}(\tau,\vec{\sigma}) = \langle\Psi\Big|\tilde{F}^{\bar{A}}_G[\mathbf{R}^{\bar{a}}(\tau,\vec{\sigma}),\mathbf{\Pi}_{\bar{a}}(\tau,\vec{\sigma})]\Big|\Psi\rangle, \quad a=1,2, \tag{8.1}$$

[60]Recall that a 3+1 splitting of the mathematical space-time, including the notions of space-like, light-like, and time-like directions, is presupposed from the beginning.

where $\mathbf{R}^{\bar{a}}(\tau,\vec{\sigma})$ and $\mathbf{\Pi}_{\bar{a}}(\tau,\vec{\sigma})$ are *scalar Dirac operators*.

Let us note that, by means of Eq.(8.1), the *non-locality* of the *classical* individuation of point-events would directly get imported at the basis of the ordinary quantum non-locality.

Also, one could evaluate in principle the expectation values of the operators corresponding to the lapse and shift functions of that gauge. Since we are considering a quantization of the 3-geometry (like in loop quantum gravity), evaluating the expectation values of the quantum 3-metric and the quantum lapse and shift functions could permit to reconstruct a coarse-grained foliation with coarse-grained WSW hyper-surfaces[61].

2) In order to avoid inequivalent Hilbert spaces, we could quantize *before* adding any gauge-fixing (i.e. independently of the choice of the 4-coordinates and the individuation of point-events), using e.g., the following rule of quantization, which respects relativistic causality: in a given canonical basis of the conjecture, quantize the two pairs of DO=BO observables and the matter DO, but leave the 8 gauge variables $\zeta^\alpha(\tau,\vec{\sigma})$, $\alpha=1,..,8$, as *c-number classical fields*. Like in Schrödinger's theory with time-dependent Hamiltonian, the momenta conjugate to the gauge variables would be represented by the functional derivatives $i\delta/\delta\zeta^\alpha(\tau,\vec{\sigma})$. Assuming that, in the chosen canonical basis of our *main conjecture*, 7 among the eight constraints be gauge momenta, we would get 7 Schrödinger equations $i\delta/\delta\zeta^\alpha(\tau,\vec{\sigma})\,\Psi(R^{\bar{a}}|\tau;\zeta^\alpha)=0$ from them. Let $H(new)\approx 0$ be the super-Hamitonian constraint and $E_{ADM}(new)$ the weak ADM energy, in the new basis. Both would become operators $\hat{H}$ *or* $\hat{E}_{ADM}(\hat{r}_a,\hat{\pi}_a,\zeta^\alpha,i\delta/\delta\zeta^\alpha)$. If an ordering existed such that the 8 quantum constraints $\hat{\phi}_\alpha$ and $\hat{E}_{ADM}$ satisfied a closed algebra $[\hat{\phi}_\alpha,\hat{\phi}_\beta]=\hat{C}_{\alpha\beta\gamma}\,\hat{\phi}_\gamma$ and $[\hat{E}_{ADM},\hat{\phi}_\alpha]=\hat{B}_{\alpha\beta}\,\hat{\phi}_\beta$ (with the quantum structure functions tending to the classical ones for $\hbar\mapsto 0$), we might quantize by imposing the following 9 coupled integrable functional Schrödinger equations

$$
\begin{aligned}
& i\frac{\delta}{\delta\zeta_\alpha(\tau,\vec{\sigma})}\,\Psi(R^{\bar{a}}|\tau;\zeta^\alpha)=0,\quad \alpha=1,..,7,\quad \Rightarrow\ \Psi=\Psi(R^{\bar{a}}|\tau;\zeta^8),\\
& \hat{H}(r^a,i\frac{\delta}{\delta r_a},\zeta^\alpha,i\frac{\delta}{\delta\zeta^\alpha})\,\Psi(R^{\bar{a}}|\tau;\zeta^\alpha)=0,\\
& i\,\frac{\partial}{\partial\tau}\,\Psi(R^{\bar{a}}|\tau;\zeta^\alpha)=\hat{E}_{ADM}(r^a,i\frac{\delta}{\delta r^a},\zeta^\alpha,i\frac{\delta}{\delta\zeta^\alpha})\,\Psi(R^{\bar{a}}|\tau;\zeta^\alpha),
\end{aligned}
\tag{8.2}
$$

with the associated usual scalar product $\langle\Psi\Big|\Psi\rangle$ being independent of τ and ζ^α's because of Eq.(8.2). This is similar to what happens in the quantization of the two-body problem in relativistic mechanics [50,51].

If the previously described quasi-Shanmugadhasan canonical basis existed, the wave functional would depend on 8 functional field parameters $\zeta^\alpha(\tau,\vec{\sigma})$, besides the mathematical time τ (actually only on ζ^8). Each *curve* in this parameter space would be associated to a Hamiltonian gauge in the following sense: for each solution Ψ of the previous equations, the classical gauge-fixings $\sigma^A - F_G^{\bar{A}}\approx 0$ implying $\zeta^\alpha=\zeta^{(G)\alpha}(R^a,\Pi_a)$, would correspond

---

[61]This foliation is called [44] the Wigner-Sen-Witten (WSW) foliation due to its properties at spatial infinity. See also footnotes 72 and 73 in Appendix B.

to expectation values $< \Psi|\, \zeta^{(G)\alpha}(\tau,\vec{\sigma})|\Psi >= \tilde{\zeta}^{(G)\alpha}(\tau,\vec{\sigma})$ defining the *curve* in the parameter space. Again, we would have a *mathematical micro space-time* and a *coarse-grained space-time of "point-events"*. At this point, by going to *coherent states*, we could try to recover classical gravitational fields[62]. The 3-geometry (volumes, areas, lengths) would be quantized, perhaps in a way coherent with the results of loop quantum gravity.

It is important to stress that, according to both of our suggestions, *only the Dirac observables would be quantized*. The upshot is that fluctuations in the gravitational field (better, in the Dirac observables) would entail fluctuations of the point texture that lends itself to the basic space-time scheme of standard relativistic quantum field theory: such fluctuating texture, however, could be recovered as a coarse-grained structure. This would induce fluctuations in the coarse-grained metrical relations, and thereby in the causal structure, both of which would tend to disappear in a semi-classical approximation. Such a situation should be conceptually tolerable, and even philosophically appealing, as compared with the impossibility of defining a causal structure within all of the attempts grounded upon quantization of the full 4-geometry.

Besides, in space-times with matter, our procedure entails *quantizing the tidal effects and action-at-a-distance potentials between matter elements but not the inertial aspects of the gravitational field*. As shown before, the latter are connected with the gauge variables whose variations reproduce all the possible viewpoints of local accelerated time-like observers. Thus, quantizing the gauge variables would be tantamount to quantizing the metric *and* the *passive observers* and their reference frames associated to the congruences studied in Section VI. Of course, such observers have nothing to do with the *dynamical observers*, which should be realized in terms of the DO of matter.

Finally, concerning different ways of looking at inertial forces, consider for the sake of completeness the few known attempts of extending non-relativistic quantum mechanics from global inertial frames to global non-inertial ones [108] by means of time-dependent unitary transformations $U(t)$. The resulting quantum potentials $V(t) = i\,\dot{U}(t)\,U^{-1}(t)$ for the fictitious forces in the new Hamiltonian $\tilde{H} = U(t)\,HU^{-1}(t) + V(t)$ for the transformed Schrödinger equation[63], as seen by an accelerated observer (passive view), are often reinterpreted as action-at-a-distance Newtonian gravitational potentials in an inertial frame (active view). This fact, implying in general a change in the emission spectra of atoms,

---

[62]At the classical level, we have the ADM Poincaré group at spatial infinity on the asymptotic Minkowski hyper-planes orthogonal to the ADM 4-momentum, while the WSW hyper-surfaces (see Appendix B) tend to such Minkowski hyper-planes in every 4-region where the 4-curvature is negligible, because their extrinsic curvature tends to zero in such regions. Thus, matter and gauge fields could be approximated there by the rest-frame relativistic fields whose quantization leads to relativistic QFT. Since at the classical level, in each 4-coordinate system, matter and gauge field satisfy $\phi(\tau,\vec{\sigma}) = \phi(\sigma^A) \approx \tilde{\phi}(F^{\bar{A}}) = \tilde{\tilde{\phi}}(R^a,\Pi_a)$, they could be thought of as functions of either the intrinsic pseudo-coordinates (as DeWitt does) or the DO=BO observables of that gauge.

[63]Note that as it happens with the time-dependent Foldy-Wouthuysen transformation [109], the operator $\tilde{H}$ describing the non-inertial time evolution is no more the energy operator.

is justified by invoking an extrapolation of the non-relativistic limit of the weak equivalence principle (universality of free fall or identity of inertial and gravitational masses) to quantum mechanics. Our Hamiltonian distinction among tidal, inertial and action-at-a-distance effects supports Synge's criticism [31]b of Einstein's statements about the equivalence of *uniform* gravitational fields and *uniform* accelerated frames. Genuine physical uniform gravitational fields do not exist over finite regions[64] and must be replaced by tidal and action-at-a-distance effects: these, however, are clearly not equivalent to uniform acceleration effects. From our point of view, the latter are generated as inertial effects whose appearance depends upon the gauge variables. Consequently, the non-relativistic limit of our quantization procedure should be consistent with the previous passive view in which atom spectra are not modified by pure inertial effects, and should match the formulation of standard non-relativistic quantum mechanics of Ref. [101].

[64] Nor is their definition a unambiguous task in general [110].

APPENDIX A: ADM CANONICAL METRIC GRAVITY.

Let M^4 be a globally hyperbolic pseudo-Riemannian 4-manifold, asymptotically flat at spatial infinity, Let M^4 be foliated (3+1 splitting or slicing) with space-like Cauchy hypersurfaces Σ_τ through the embeddings $i_\tau : \Sigma \rightarrow \Sigma_\tau \subset M^4$, $\vec{\sigma} \mapsto x^\mu = z^\mu(\tau, \vec{\sigma})$, of a 3-manifold Σ, assumed diffeomorphic to R^3, into M^4 [65]. The non-degenerate 4-metric tensor ${}^4g_{\mu\nu}(x)$ has Lorentzian signature $\epsilon(+,-,-,-)$ [66].

Let $n^\mu(\sigma)$ and $l^\mu(\sigma) = N(\sigma) n^\mu(\sigma)$ be the controvariant timelike normal and unit normal $[{}^4g_{\mu\nu}(z(\sigma))\, l^\mu(\sigma) l^\nu(\sigma) = \epsilon]$ to Σ_τ at the point $z(\sigma) \in \Sigma_\tau$. The positive function $N(\sigma) > 0$ is the lapse function: $N(\sigma) d\tau$ measures the proper time interval at $z(\sigma) \in \Sigma_\tau$ between Σ_τ and $\Sigma_{\tau+d\tau}$. The shift functions $N^r(\sigma)$ are defined so that $N^r(\sigma) d\tau$ describes the horizontal shift on Σ_τ such that, if $z^\mu(\tau + d\tau, \vec{\sigma} + d\vec{\sigma}) \in \Sigma_{\tau+d\tau}$, then $z^\mu(\tau + d\tau, \vec{\sigma} + d\vec{\sigma}) \approx z^\mu(\tau, \vec{\sigma}) + N(\tau, \vec{\sigma}) d\tau l^\mu(\tau, \vec{\sigma}) + [d\sigma^r + N^r(\tau, \vec{\sigma}) d\tau] \frac{\partial z^\mu(\tau, \vec{\sigma})}{\partial \sigma^r}$; therefore, the so called evolution vector is $\frac{\partial z^\mu(\sigma)}{\partial \tau} = N(\sigma) l^\mu(\sigma) + N^r(\sigma) \frac{\partial z^\mu(\tau, \vec{\sigma})}{\partial \sigma^r}$. The covariant unit normal to Σ_τ is $l_\mu(\sigma) = {}^4g_{\mu\nu}(z(\sigma)) l^\nu(\sigma) = N(\sigma) \partial_\mu \tau |_{x=z(\sigma)}$, with $\tau = \tau(\sigma)$ a global timelike future-oriented function.

Instead of local coordinates x^μ for M^4, we use coordinates σ^A on $R \times \Sigma \approx M^4$ $[x^\mu = z^\mu(\sigma)$ with inverse $\sigma^A = \sigma^A(x)]$, and the associated Σ_τ-adapted holonomic coordinate basis $\partial_A = \frac{\partial}{\partial \sigma^A} \in T(R \times \Sigma) \mapsto b^\mu_A(\sigma) \partial_\mu = \frac{\partial z^\mu(\sigma)}{\partial \sigma^A} \partial_\mu \in TM^4$ for vector fields, and $dx^\mu \in T^* M^4 \mapsto d\sigma^A = b^A_\mu(\sigma) dx^\mu = \frac{\partial \sigma^A(z)}{\partial z^\mu} dx^\mu \in T^*(R \times \Sigma)$ for differential one-forms.

In the new basis, the induced 4-metric becomes

$$
{}^4g_{AB} = \begin{pmatrix} {}^4g_{\tau\tau} = \epsilon (N^2 - {}^3g_{rs} N^r N^s) & {}^4g_{\tau s} = -\epsilon\, {}^3g_{su} N^u \\ {}^4g_{\tau r} = -\epsilon\, {}^3g_{rv} N^v & {}^4g_{rs} = -\epsilon\, {}^3g_{rs} \end{pmatrix}, \tag{A1}
$$

where the 3-metric ${}^3g_{rs} = -\epsilon\, {}^4g_{rs}$ with signature $(+++)$, of Σ_τ has been introduced. The line element of M^4 is

$$
\begin{aligned}
ds^2 &= {}^4g_{\mu\nu} dx^\mu dx^\nu = \epsilon (N^2 - {}^3g_{rs} N^r N^s)(d\tau)^2 - 2\epsilon\, {}^3g_{rs} N^s d\tau d\sigma^r - \epsilon\, {}^3g_{rs} d\sigma^r d\sigma^s = \\
&= \epsilon \Big[N^2 (d\tau)^2 - {}^3g_{rs} (d\sigma^r + N^r d\tau)(d\sigma^s + N^s d\tau) \Big],
\end{aligned} \tag{A2}
$$

such that $\epsilon\, {}^4g_{oo} > 0$, $\epsilon\, {}^4g_{ij} < 0$, $\begin{vmatrix} {}^4g_{ii} & {}^4g_{ij} \\ {}^4g_{ji} & {}^4g_{jj} \end{vmatrix} > 0$, $\epsilon\, det\, {}^4g_{ij} > 0$, hold true.

Defining $g = {}^4g = |\, det\, ({}^4g_{\mu\nu})\, |$ and $\gamma = {}^3g = |\, det\, ({}^3g_{rs})\, |$, the lapse and shift functions assume the form

---

[65] $\tau : M^4 \rightarrow R$ is a global, time-like, future-oriented function labelling the leaves of the foliation; x^μ are local coordinates in a chart of M^4; $\vec{\sigma} = \{\sigma^r\}$, r=1,2,3, are coordinates in a global chart of Σ, which is diffeomorphic to R^3; we shall use the notations $\sigma^A = (\sigma^\tau = \tau; \vec{\sigma})$, $A = \tau, r$, for the coordinates of M^4 adapted to the 3+1 splitting, and $z^\mu(\sigma) = z^\mu(\tau, \vec{\sigma})$ for the embedding functions.

[66] Here $\epsilon = \pm 1$ according to particle physics and general relativity conventions, respectively.

$$
\begin{aligned}
&N=\sqrt{\frac{{}^4g}{{}^3g}}=\frac{1}{\sqrt{{}^4g^{\tau\tau}}}=\sqrt{\frac{g}{\gamma}}=\sqrt{{}^4g_{\tau\tau}-\epsilon\,{}^3g^{rs}\,{}^4g_{\tau r}{}^4g_{\tau s}},\\
&N^r=-\epsilon\,{}^3g^{rs}\,{}^4g_{\tau s}=-\frac{{}^4g^{\tau r}}{{}^4g^{\tau\tau}},\quad N_r={}^3g_{rs}N^s=-\epsilon\;{}^4g_{rs}N^s=-\epsilon{}^4g_{\tau r}.
\end{aligned}
\tag{A3}
$$

Given an arbitrary 3+1 splitting of M^4, the ADM action [43] expressed in terms of the independent Σ_τ-adapted variables N, $N_r={}^3g_{rs}N^s$, ${}^3g_{rs}$ is

$$
\begin{aligned}
S_{ADM}&=\int d\tau\,L_{ADM}(\tau)=\int d\tau d^3\sigma\mathcal{L}_{ADM}(\tau,\vec{\sigma})=\\
&=-\epsilon k\int_{\triangle\tau}d\tau\int d^3\sigma\,\{\sqrt{\gamma}N\,[{}^3R+{}^3K_{rs}\,{}^3K^{rs}-({}^3K)^2]\}(\tau,\vec{\sigma}),
\end{aligned}
\tag{A4}
$$

where $k=\frac{c^3}{16\pi G}$, with G the Newton constant.

The Euler-Lagrange equations are[67]

$$
\begin{aligned}
L_N&=\frac{\partial\mathcal{L}_{ADM}}{\partial N}-\partial_\tau\frac{\partial\mathcal{L}_{ADM}}{\partial\partial_\tau N}-\partial_r\frac{\partial\mathcal{L}_{ADM}}{\partial\partial_r N}=\\
&=-\epsilon k\sqrt{\gamma}[{}^3R-{}^3K_{rs}\,{}^3K^{rs}+({}^3K)^2]=-2\epsilon k\,{}^4\bar{G}_{ll}\overset{\circ}{=}0,\\
L^r_{\vec{N}}&=\frac{\partial\mathcal{L}_{ADM}}{\partial N_r}-\partial_\tau\frac{\partial\mathcal{L}_{ADM}}{\partial\partial_\tau N_r}-\partial_s\frac{\partial\mathcal{L}_{ADM}}{\partial\partial_s N_r}=\\
&=2\epsilon k[\sqrt{\gamma}({}^3K^{rs}-{}^3g^{rs}\,{}^3K)]_{|s}=2k\,{}^4\bar{G}_l{}^r\overset{\circ}{=}0,\\
L^{rs}_g&=-\epsilon k\Big[\frac{\partial}{\partial\tau}[\sqrt{\gamma}({}^3K^{rs}-{}^3g^{rs}\,{}^3K)]-N\sqrt{\gamma}({}^3R^{rs}-\frac{1}{2}{}^3g^{rs}\,{}^3R)+\\
&\quad+2N\sqrt{\gamma}({}^3K^{ru}\,{}^3K_u{}^s-{}^3K\,{}^3K^{rs})+\frac{1}{2}N\sqrt{\gamma}[({}^3K)^2-{}^3K_{uv}\,{}^3K^{uv})^3g^{rs}+\\
&\quad+\sqrt{\gamma}({}^3g^{rs}N^{|u}{}_{|u}-N^{|r|s})\Big]=-\epsilon kN\sqrt{\gamma}\,{}^4\bar{G}^{rs}\overset{\circ}{=}0,
\end{aligned}
\tag{A5}
$$

and correspond to Einstein's equations in the form ${}^4\bar{G}_{ll}\overset{\circ}{=}0$, ${}^4\bar{G}_{lr}\overset{\circ}{=}0$, ${}^4\bar{G}_{rs}\overset{\circ}{=}0$, respectively. The four contracted Bianchi identities imply that only two of the six equations $L^{rs}_g\overset{\circ}{=}0$ are independent.

The canonical momenta (densities of weight -1) are

$$
\begin{aligned}
&\tilde{\pi}^N(\tau,\vec{\sigma})=\frac{\delta S_{ADM}}{\delta\partial_\tau N(\tau,\vec{\sigma})}=0,\\
&\tilde{\pi}^r_{\vec{N}}(\tau,\vec{\sigma})=\frac{\delta S_{ADM}}{\delta\partial_\tau N_r(\tau,\vec{\sigma})}=0,\\
&{}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma})=\frac{\delta S_{ADM}}{\delta\partial_\tau{}^3g_{rs}(\tau,\vec{\sigma})}=\epsilon k\,[\sqrt{\gamma}({}^3K^{rs}-{}^3g^{rs}\,{}^3K)](\tau,\vec{\sigma}),
\end{aligned}
$$

$\Downarrow$

---

[67] The symbol $\overset{\circ}{=}$ means "evaluated on the extremals of the variational principle", namely on the solutions of the equation of motion.

$$ {}^3K_{rs} = \frac{\epsilon}{k\sqrt{\gamma}}[{}^3\tilde{\Pi}_{rs} - \frac{1}{2}{}^3g_{rs}\,{}^3\tilde{\Pi}], \qquad {}^3\tilde{\Pi} = {}^3g_{rs}\,{}^3\tilde{\Pi}^{rs} = -2\epsilon k\sqrt{\gamma}\,{}^3K, \tag{A6} $$

and satisfy the Poisson brackets

$$ \begin{aligned} &\{N(\tau,\vec{\sigma}), \tilde{\Pi}^N(\tau,\vec{\sigma}^{'})\} = \delta^3(\vec{\sigma},\vec{\sigma}^{'}),\\ &\{N_r(\tau,\vec{\sigma}), \tilde{\Pi}^s_{\vec{N}}(\tau,\vec{\sigma}^{'})\} = \delta^s_r\delta^3(\vec{\sigma},\vec{\sigma}^{'}),\\ &\{{}^3g_{rs}(\tau,\vec{\sigma}), {}^3\tilde{\Pi}^{uv}(\tau,\vec{\sigma}^{'}\} = \frac{1}{2}(\delta^u_r\delta^v_s + \delta^v_r\delta^u_s)\delta^3(\vec{\sigma},\vec{\sigma}^{'}). \end{aligned} \tag{A7} $$

The Wheeler- De Witt super-metric is

$$ {}^3G_{rstw}(\tau,\vec{\sigma}) = [{}^3g_{rt}\,{}^3g_{sw} + {}^3g_{rw}\,{}^3g_{st} - {}^3g_{rs}\,{}^3g_{tw}](\tau,\vec{\sigma}). \tag{A8} $$

Its inverse is defined by the equations

$$ \begin{aligned} \frac{1}{2}{}^3G_{rstw}\,\frac{1}{2}{}^3G^{twuv} &= \frac{1}{2}(\delta^u_r\delta^v_s + \delta^v_r\delta^u_s),\\ {}^3G^{twuv}(\tau,\vec{\sigma}) &= [{}^3g^{tu}\,{}^3g^{wv} + {}^3g^{tv}\,{}^3g^{wu} - 2\,{}^3g^{tw}\,{}^3g^{uv}](\tau,\vec{\sigma}), \end{aligned} \tag{A9} $$

so that

$$ \begin{aligned} {}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma}) &= \frac{1}{2}\epsilon k\sqrt{\gamma}\,{}^3G^{rsuv}(\tau,\vec{\sigma})\,{}^3K_{uv}(\tau,\vec{\sigma}),\\ {}^3K_{rs}(\tau,\vec{\sigma}) &= \frac{\epsilon}{2k\sqrt{\gamma}}\,{}^3G_{rsuv}(\tau,\vec{\sigma})\,{}^3\tilde{\Pi}^{uv}(\tau,\vec{\sigma}),\\ \partial_\tau\,{}^3g_{rs}(\tau,\vec{\sigma}) &= [N_{r|s} + N_{s|r} - \frac{\epsilon N}{k\sqrt{\gamma}}\,{}^3G_{rsuv}\,{}^3\tilde{\Pi}^{uv}](\tau,\vec{\sigma}). \end{aligned} \tag{A10} $$

Since ${}^3\tilde{\Pi}^{rs}\partial_\tau\,{}^3g_{rs} = {}^3\tilde{\Pi}^{rs}[N_{r|s} + N_{s|r} - \frac{\epsilon N}{k\sqrt{\gamma}}{}^3G_{rsuv}{}^3\tilde{\Pi}^{uv}] = -2N_r{}^3\tilde{\Pi}^{rs}{}_{|s} - \frac{\epsilon N}{k\sqrt{\gamma}}\,{}^3G_{rsuv}\,{}^3\tilde{\Pi}^{rs3}\tilde{\Pi}^{uv} + (2N_r\,{}^3\tilde{\Pi}^{rs})_{|s}$, we obtain the canonical Hamiltonian [68]

$$ \begin{aligned} H_{(c)ADM} &= \int_S d^3\sigma\,[\tilde{\pi}^N\partial_\tau N + \tilde{\pi}^r_{\vec{N}}\partial_\tau N_r + {}^3\tilde{\Pi}^{rs}\partial_\tau{}^3g_{rs}](\tau,\vec{\sigma}) - L_{ADM} =\\ &= \int_S d^3\sigma\,[\epsilon N(k\sqrt{\gamma}\,{}^3R - \frac{1}{2k\sqrt{\gamma}}{}^3G_{rsuv}{}^3\tilde{\Pi}^{rs3}\tilde{\Pi}^{uv}) - 2N_r\,{}^3\tilde{\Pi}^{rs}{}_{|s}](\tau,\vec{\sigma}) +\\ &+ 2\int_{\partial S} d^2\Sigma_s[N_r\,{}^3\tilde{\Pi}^{rs}\;](\tau,\vec{\sigma}). \end{aligned} \tag{A11} $$

In the following discussion the surface term will be omitted.

The Dirac Hamiltonian is [69]

---

[68] Since $N_r\,{}^3\tilde{\Pi}^{rs}$ is a vector density of weight -1, it holds ${}^3\nabla_s(N_r\,{}^3\tilde{\Pi}^{rs}) = \partial_s(N_r\,{}^3\tilde{\Pi}^{rs})$.

[69] The $\lambda(\tau,\vec{\sigma})$'s are arbitrary Dirac multipliers.

$$H_{(D)ADM} = H_{(c)ADM} + \int d^3\sigma \, [\lambda_N \, \tilde{\pi}^N + \lambda_r^{\vec{N}} \, \tilde{\pi}^r_{\vec{N}}](\tau, \vec{\sigma}). \tag{A12}$$

The τ-constancy of the primary constraints $[\partial_\tau \tilde{\pi}^N(\tau,\vec{\sigma}) \stackrel{\circ}{=} \{\tilde{\pi}^N(\tau,\vec{\sigma}), H_{(D)ADM}\} \approx 0$, $\partial_\tau \tilde{\pi}^r_{\vec{N}}(\tau,\vec{\sigma}) \stackrel{\circ}{=} \{\tilde{\pi}^r_{\vec{N}}(\tau,\vec{\sigma}), H_{(D)ADM}\} \approx 0]$ generates four secondary constraints (densities of weight -1) which correspond to the Einstein equations ${}^4\bar{G}_{ll}(\tau,\vec{\sigma}) \stackrel{\circ}{=} 0$, ${}^4\bar{G}_{lr}(\tau,\vec{\sigma}) \stackrel{\circ}{=} 0$

$$\begin{aligned}
\tilde{\mathcal{H}}(\tau,\vec{\sigma}) &= \epsilon[k\sqrt{\gamma}\,{}^3R - \frac{1}{2k\sqrt{\gamma}}{}^3G_{rsuv}\,{}^3\tilde{\Pi}^{rs}\,{}^3\tilde{\Pi}^{uv}](\tau,\vec{\sigma}) = \\
&= \epsilon[\sqrt{\gamma}\,{}^3R - \frac{1}{k\sqrt{\gamma}}({}^3\tilde{\Pi}^{rs}\,{}^3\tilde{\Pi}_{rs} - \frac{1}{2}({}^3\tilde{\Pi})^2)](\tau,\vec{\sigma}) \approx 0, \\
{}^3\tilde{\mathcal{H}}^r(\tau,\vec{\sigma}) &= -2\,{}^3\tilde{\Pi}^{rs}{}_{|s}(\tau,\vec{\sigma}) = -2[\partial_s\,{}^3\tilde{\Pi}^{rs} + {}^3\Gamma^r_{su}\,{}^3\tilde{\Pi}^{su}](\tau,\vec{\sigma}) = \\
&= -2\epsilon k\{\partial_s[\sqrt{\gamma}({}^3K^{rs} - {}^3g^{rs}\,{}^3K)] + {}^3\Gamma^r_{su}\sqrt{\gamma}({}^3K^{su} - {}^3g^{su}\,{}^3K)\}(\tau,\vec{\sigma}) \approx 0,
\end{aligned} \tag{A13}$$

so that the Hamiltonian becomes

$$H_{(c)ADM} = \int d^3\sigma[N\,\tilde{\mathcal{H}} + N_r\,{}^3\tilde{\mathcal{H}}^r](\tau,\vec{\sigma}) \approx 0, \tag{A14}$$

with $\tilde{\mathcal{H}}(\tau,\vec{\sigma}) \approx 0$ called the superhamiltonian constraint and ${}^3\tilde{\mathcal{H}}^r(\tau,\vec{\sigma}) \approx 0$ the supermomentum constraints. One may say that the term $-\epsilon k\sqrt{\gamma}({}^3K_{rs}\,{}^3K^{rs} - {}^3K^2)$ in $\tilde{\mathcal{H}}(\tau,\vec{\sigma}) \approx 0$, is the kinetic energy and $\epsilon k\sqrt{\gamma}\,{}^3R$ the potential energy.

All the constraints are first class, because the only non-identically zero Poisson brackets correspond to the so called universal Dirac algebra [45]:

$$\begin{aligned}
\{{}^3\tilde{\mathcal{H}}_r(\tau,\vec{\sigma}), {}^3\tilde{\mathcal{H}}_s(\tau,\vec{\sigma}^{'})\} &= {}^3\tilde{\mathcal{H}}_r(\tau,\vec{\sigma}^{'})\,\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^s} + {}^3\tilde{\mathcal{H}}_s(\tau,\vec{\sigma})\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^r}, \\
\{\tilde{\mathcal{H}}(\tau,\vec{\sigma}), {}^3\tilde{\mathcal{H}}_r(\tau,\vec{\sigma}^{'})\} &= \tilde{\mathcal{H}}(\tau,\vec{\sigma})\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^r}, \\
\{\tilde{\mathcal{H}}(\tau,\vec{\sigma}), \tilde{\mathcal{H}}(\tau,\vec{\sigma}^{'})\} &= [{}^3g^{rs}(\tau,\vec{\sigma}){}^3\tilde{\mathcal{H}}_s(\tau,\vec{\sigma}) + \\
&+ {}^3g^{rs}(\tau,\vec{\sigma}^{'}){}^3\tilde{\mathcal{H}}_s(\tau,\vec{\sigma}^{'})]\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^r},
\end{aligned} \tag{A15}$$

with ${}^3\tilde{\mathcal{H}}_r = {}^3g_{rs}\,{}^3\tilde{\mathcal{H}}^r$ as the combination of the supermomentum constraints satisfying the algebra of 3-diffeomorphisms. In Ref. [52] it is shown that Eqs.(A15) are sufficient conditions for the embeddability of Σ_τ into M^4. In the second paper in Ref. [111] it is shown that the last two lines of the Dirac algebra are the phase space equivalent of the Bianchi's identities ${}^4G^{\mu\nu}{}_{;\nu} \equiv 0$.

The Hamilton-Dirac equations are [$\mathcal{L}$ is the notation for the Lie derivative]

$$\begin{aligned}
\partial_\tau N(\tau,\vec{\sigma}) &\stackrel{\circ}{=} \{N(\tau,\vec{\sigma}), H_{(D)ADM}\} = \lambda_N(\tau,\vec{\sigma}), \\
\partial_\tau N_r(\tau,\vec{\sigma}) &\stackrel{\circ}{=} \{N_r(\tau,\vec{\sigma}), H_{(D)ADM}\} = \lambda_r^{\vec{N}}(\tau,\vec{\sigma}), \\
\partial_\tau\,{}^3g_{rs}(\tau,\vec{\sigma}) &\stackrel{\circ}{=} \{{}^3g_{rs}(\tau,\vec{\sigma}), H_{(D)ADM}\} = [N_{r|s} + N_{s|r} - \frac{2\epsilon N}{k\sqrt{\gamma}}({}^3\tilde{\Pi}_{rs} - \frac{1}{2}{}^3g_{rs}\,{}^3\tilde{\Pi})](\tau,\vec{\sigma}) = \\
&= [N_{r|s} + N_{s|r} - 2N\,{}^3K_{rs}](\tau,\vec{\sigma}),
\end{aligned}$$

$$
\begin{aligned}
\partial_\tau\, {}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma}) \,&{\buildrel \circ \over =}\, \{{}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma}), H_{(D)ADM}\} = \epsilon[N\, k\sqrt{\gamma}({}^3R^{rs} - \frac{1}{2}{}^3g^{rs}\, {}^3R)](\tau,\vec{\sigma}) -\\
&- 2\epsilon[\frac{N}{k\sqrt{\gamma}}(\frac{1}{2}{}^3\tilde{\Pi}\, {}^3\tilde{\Pi}^{rs} - {}^3\tilde{\Pi}^r{}_u\, {}^3\tilde{\Pi}^{us})(\tau,\vec{\sigma}) -\\
&- \frac{\epsilon N}{2}\frac{{}^3g^{rs}}{k\sqrt{\gamma}}(\frac{1}{2}{}^3\tilde{\Pi}^2 - {}^3\tilde{\Pi}_{uv}\, {}^3\tilde{\Pi}^{uv})](\tau,\vec{\sigma}) +\\
&+ \mathcal{L}_{\vec{N}}\, {}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma}) + \epsilon[k\sqrt{\gamma}(N^{|r|s} - {}^3g^{rs}\, N^{|u}{}_{|u})](\tau,\vec{\sigma}),\\
&\\
&\Downarrow\\
&\\
\partial_\tau\, {}^3K_{rs}(\tau,\vec{\sigma}) \,&{\buildrel \circ \over =}\, \Big(N[{}^3R_{rs} + {}^3K\, {}^3K_{rs} - 2\, {}^3K_{ru}\, {}^3K^u{}_s] -\\
&- N_{|s|r} + N^u{}_{|s}\, {}^3K_{ur} + N^u{}_{|r}\, {}^3K_{us} + N^u\, {}^3K_{rs|u}\Big)(\tau,\vec{\sigma}),\\
&\\
&with\\
\mathcal{L}_{\vec{N}}\, {}^3\tilde{\Pi}^{rs} &= -\sqrt{\gamma}\, {}^3\nabla_u(\frac{N^u}{\sqrt{\gamma}}{}^3\tilde{\Pi}^{rs}) + {}^3\tilde{\Pi}^{ur}\, {}^3\nabla_u N^s + {}^3\tilde{\Pi}^{us}\, {}^3\nabla_u N^r.
\end{aligned}
\qquad (A16)
$$

The above equation for $\partial_\tau\, {}^3g_{rs}(\tau,\vec{\sigma})$ shows that the generator of space pseudo-diffeomorphisms [70] $\int d^3\sigma N_r(\tau,\vec{\sigma})\ \ {}^3\tilde{\mathcal{H}}^r(\tau,\vec{\sigma})$ produces a variation, tangent to Σ_τ, $\delta_{tangent}{}^3g_{rs} = \mathcal{L}_{\vec{N}}\, {}^3g_{rs} = N_{r|s} + N_{s|r}$ in accord with the infinitesimal pseudo-diffeomorphisms in $Diff\, \Sigma_\tau$. Instead, the gauge transformations induced by the super-hamiltonian generator $\int d^3\sigma N(\tau,\vec{\sigma})\ \tilde{\mathcal{H}}(\tau,\vec{\sigma})$ do not reproduce the infinitesimal diffeomorphisms in $Diff\, M^4$ normal to Σ_τ (see Ref. [111]). For a further clarification of the connection between space-time diffeomorphisms and Hamiltonian gauge transformations see Ref. [44].

[70] The Hamiltonian transformations generated by these constraints are the extension to the 3-metric of passive diffeomorphisms, namely changes of coordinate charts, of Σ_τ $[Diff\, \Sigma_\tau]$.

APPENDIX B: CONGRUENCES OF TIME-LIKE ACCELERATED OBSERVERS.

In this Appendix we consider the point of view of the special (non-rotating, surface-forming) congruence of time-like accelerated observers whose 4-velocity field is the field of unit normals to the space-like hyper-surfaces Σ_τ; as evolution parameter of the Hamiltonian description we use the parameter τ, labeling the leaves of the foliation[71].

We want to describe this non-rotating Hamiltonian congruence, by emphasizing its interpretation in terms of gauge variables and DO. The field of contravariant and covariant unit normals to the space-like hyper-surfaces Σ_τ are expressed only in terms of the lapse and shift gauge variables (as in Sections III and IV, we use coordinates adapted to the foliation: $l^A(\tau,\vec{\sigma}) = b^A_\mu(\tau,\vec{\sigma})\, l^\mu(\tau,\vec{\sigma})$ with the $b^A_\mu(\tau,\vec{\sigma}) = \frac{\partial \sigma^A}{\partial z^\mu}$ being the transition coefficients from adapted to general coordinates)

$$
\begin{aligned}
l^A(\tau,\vec{\sigma}) &= \frac{1}{N(\tau,\vec{\sigma})}\Big(1; -N^r(\tau,\vec{\sigma})\Big),\\
l_A(\tau,\vec{\sigma}) &= N(\tau,\vec{\sigma})\Big(1;0\Big), \qquad l^A(\tau,\vec{\sigma})\, l_A(\tau,\vec{\sigma}) = 1.
\end{aligned}
\tag{B1}
$$

Since this congruence is surface forming by construction, it has zero vorticity and is non-rotating (in the sense of congruences). According to footnote 18, in Christodoulou-Klainermann spacetimes we have $N(\tau,\vec{\sigma}) = \epsilon + n(\tau,\vec{\sigma})$, $N^r(\tau,\vec{\sigma}) = n^r(\tau,\vec{\sigma})$. The specific time-like direction identified by the normal has inertial-like nature, in the sense of being dependent on Hamiltonian gauge variables only. Therefore the world-lines of the observers of this foliation[72] change on-shell going from a 4-coordinate system to another. On the other hand, the embeddings $z^\mu_\Sigma(\tau,\vec{\sigma})$ of the leaves Σ_τ of the WSW foliation in space-time[73] depend on *both* the DO and the gauge variables.

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[71]According to Ref. [86], this is the *hyper-surface point of view*. The *threading point of view* is instead a description involving only a rotating congruence of observers: since the latter is rotating, it is not surface-forming (non-zero vorticity) and in each point we can only divide the tangent space in the direction parallel to the 4-velocity and the orthogonal complement (the local rest frame). On the other hand, the *slicing point of view*, originally adopted in ADM canonical gravity, uses two congruences: the non-rotating one with the normals to Σ_τ as 4-velocity fields and a second (rotating, non-surface-forming) congruence of observers, whose 4-velocity field is the field of time-like unit vectors determined by the τ derivative of the embeddings identifying the leaves Σ_τ (their so-called evolution vector field). Furthermore, as Hamiltonian evolution parameter it uses the affine parameter describing the world-lines of this second family of observers.

[72]It is called the Wigner-Sen-Witten (WSW) foliation [44] due to its properties at spatial infinity. The associated observers are called *Eulerian observers* when a perfect fluid is present as dynamical matter.

[73]See Ref. [44], Eqs.(12.8) and (12.9), for the way in which the embedding of the WSW foliation corresponding to a solution of Einstein equations has to be determined in every 4-coordinate system corresponding on-shell to a completely fixed Hamiltonian gauge. As said in footnote 18, there are asymptotic inertial observers corresponding to the *fixed stars* that can be endowed with a spatial

[74] If $x^{\mu}_{\vec{\sigma}_o}(\tau)$ is the time-like world-line of the observer crossing the leave Σ_{τ_o} at $\vec{\sigma}_o$, we have

$$
\begin{aligned}
&x^{\mu}_{\vec{\sigma}_o}(\tau) = z^{\mu}_{\Sigma}(\tau, \vec{\rho}_{\vec{\sigma}_o}(\tau)), \quad with\ \vec{\rho}_{\vec{\sigma}_o}(\tau_o) = \vec{\sigma}_o, \qquad \dot{x}^{\mu}_{\vec{\sigma}_o}(\tau) = \frac{dx^{\mu}_{\vec{\sigma}_o}(\tau)}{d\tau},\\
&l^{\mu}_{\vec{\sigma}_o}(\tau) = l^{\mu}(\tau, \vec{\rho}_{\vec{\sigma}_o}(\tau)) = \frac{\dot{x}^{\mu}_{\vec{\sigma}_o}(\tau)}{\sqrt{{}^4g_{\alpha\beta}(x_{\vec{\sigma}_o}(\tau))\, \dot{x}^{\alpha}_{\vec{\sigma}_o}(\tau)\, \dot{x}^{\beta}_{\vec{\sigma}_o}(\tau)}},\\
&\qquad a^{\mu}_{\vec{\sigma}_o}(\tau) = \frac{dl^{\mu}_{\vec{\sigma}_o}(\tau)}{d\tau}, \qquad a^{\mu}_{\vec{\sigma}_o}(\tau)\, l_{\vec{\sigma}_o\,\mu}(\tau) = 0.
\end{aligned}
\tag{B2}
$$

Here $a^{\mu}_{\vec{\sigma}_o}(\tau)$ is the 4-acceleration of the observer $x^{\mu}_{\vec{\sigma}_o}(\tau)$.

As for any congruence, we have the decomposition ($P_{\mu\nu} = \eta_{\mu\nu} - l_{\mu}\, l_{\nu}$)

$$
\begin{aligned}
{}^4\nabla_{\mu}\, l_{\nu} &= l_{\mu}\, a_{\nu} + \frac{1}{3}\, \Theta\, P_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu},\\
&a^{\mu} = l^{\nu}\, {}^4\nabla_{\nu}\, l^{\mu} = \dot{l}^{\mu},\\
&\Theta = {}^4\nabla_{\mu}\, l^{\mu},\\
&\sigma_{\mu\nu} = \frac{1}{2}\, (a_{\mu}\, l_{\nu} + a_{\nu}\, l_{\mu}) + \frac{1}{2}\, ({}^4\nabla_{\mu}\, l_{\nu} + {}^4\nabla_{\nu}\, l_{\mu}) - \frac{1}{3}\, \Theta\, P_{\mu\nu},\\
&withmagnitude \sigma^2 = \frac{1}{2} \sigma_{\mu\nu} \sigma^{\mu\nu}),\\
&\omega_{\mu\nu} = -\omega_{\nu\mu} = \epsilon_{\mu\nu\alpha\beta}\, \omega^{\alpha}\, l^{\beta} = \frac{1}{2}\, (a_{\mu}\, l_{\nu} - a_{\nu}\, l_{\mu}) + \frac{1}{2}\, ({}^4\nabla_{\mu}\, l_{\nu} - {}^4\nabla_{\nu}\, l_{\mu}) = 0,\\
&\omega^{\mu} = \frac{1}{2}\, \epsilon^{\mu\alpha\beta\gamma}\, \omega_{\alpha\beta}\, l_{\gamma} = 0,
\end{aligned}
\tag{B3}
$$

where a^{μ} is the 4-acceleration, Θ the *expansion* (it measures the average expansion of the infinitesimally nearby world-lines surrounding a given world-line in the congruence), $\sigma_{\mu\nu}$ the *shear* (it measures how an initial sphere in the tangent space to the given world-line, which is Lie transported along l^{μ} [75], is distorted towards an ellipsoid with principal axes given by the eigenvectors of ${\sigma^{\mu}}_{\nu}$, with rate given by the eigenvalues of ${\sigma^{\mu}}_{\nu}$) and $\omega_{\mu\nu}$ the *twist*

---

triad. Then the asymptotic tetrad formed by the ADM 4-momentum and the spatial triad can be transported in a dynamical way (on-shell) by using the Sen-Witten connection [113] in the Frauendiener formulation [114] in every point of Σ_{τ}. This defines a *local compass of inertia* to be compared with the local gyroscopes (whether Fermi-Walker transported or not). The WSW local compass of inertia consists in pointing to the fixed stars with a telescope. It is needed in a satellite like Gravity Probe B to detect the frame dragging (or gravitomagnetic Lense-Thirring effect) of the inertial frames by means of the rotation of a FW transported gyroscope relative to it.

[74] Note that the mathematical time parameter τ labeling the leaves of the foliation is not in general the proper time of any observer of the congruence.

[75] It has zero Lie derivative with respect to $l^{\mu}\, \partial_{\mu}$.

or vorticity (it measures the rotation of the nearby world-lines infinitesimally surrounding the given one); $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ are purely spatial ($\sigma_{\mu\nu} l^\nu = \omega_{\mu\nu} l^\nu = 0$). Due to the Frobenius theorem, the congruence is (locally) hyper-surface orthogonal if and only if $\omega_{\mu\nu} = 0$. The equation $\frac{1}{l}\, l^\mu\, \partial_\mu\, l = \frac{1}{3}\,\Theta$ defines a representative length l along the world-line of l^μ, describing the volume expansion (or contraction) behaviour of the congruence.

While all these quantities depend on the Hamiltonian gauge variables, the expansion and the shear depend a priori also upon the DO, because the covariant derivative is used in their definition.

Yet, the ADM canonical formalism provides *additional information*. Actually, on each space-like hyper-surface Σ_τ of the foliation, there is a *privileged contravariant space-like direction* identified by the lapse and shift gauge variables [76]

$$\begin{aligned}
\mathcal{N}^\mu(\tau,\vec{\sigma}) &= \frac{1}{|\vec{N}(\tau,\vec{\sigma})|}\,\Big(0; n^r(\tau,\vec{\sigma})\Big),\\
\mathcal{N}_\mu(\tau,\vec{\sigma}) &= |\vec{N}(\tau,\vec{\sigma})|\,\Big(1; \frac{N_r(\tau,\vec{\sigma})}{|\vec{N}(\tau,\vec{\sigma})|^2}\Big),\\
&\mathcal{N}^\mu(\tau,\vec{\sigma})\, l_\mu(\tau,\vec{\sigma}) = 0, \qquad \mathcal{N}^\mu(\tau,\vec{\sigma})\,\mathcal{N}_\mu(\tau,\vec{\sigma}) = -1,\\
&|\vec{N}(\tau,\vec{\sigma})| = \sqrt{({}^3g_{rs}\, N^r\, N^s)(\tau,\vec{\sigma})}.
\end{aligned} \tag{B4}$$

If 4-coordinates, corresponding to an on-shell complete Hamiltonian gauge fixing, exist such that the vector field defined by $\mathcal{N}^\mu(\tau,\vec{\sigma})$ on each Σ_τ is surface-forming (zero vorticity[77]), then each Σ_τ can be foliated with 2-surfaces, and the 3+1 splitting of space-time becomes a (2+1)+1 splitting corresponding to the 2+2 splittings studied by Stachel and d'Inverno [116].

We have therefore a natural candidate for *one* of the three spatial vectors of each observer, namely: $E^\mu_{\vec{\sigma}_o\,(\mathcal{N})}(\tau) = \mathcal{N}^\mu_{\vec{\sigma}_o}(\tau) = \mathcal{N}^\mu(\tau, \vec{\rho}_{\vec{\sigma}_o}(\tau))$. By means of $l^\mu_{\vec{\sigma}_o}(\tau) = l^\mu(\tau, \vec{\rho}_{\vec{\sigma}_o}(\tau))$ and $\mathcal{N}^\mu_{\vec{\sigma}_o}(\tau)$, we can construct two *null vectors* at each space-time point

$$\begin{aligned}
\mathcal{K}^\mu_{\vec{\sigma}_o}(\tau) &= \sqrt{\frac{|\vec{N}|}{2}}\,\Big(l^\mu_{\vec{\sigma}_o}(\tau) + \mathcal{N}^\mu_{\vec{\sigma}_o}(\tau)\Big),\\
\mathcal{L}^\mu_{\vec{\sigma}_o}(\tau) &= \frac{1}{\sqrt{2\,|\vec{N}|}}\,\Big(l^\mu_{\vec{\sigma}_o}(\tau) - \mathcal{N}^\mu_{\vec{\sigma}_o}(\tau)\Big).
\end{aligned} \tag{B5}$$

and then get a *null tetrad* of the type used in the Newman-Penrose formalism [38]. The last two axes of the spatial triad can be chosen as two space-like circular complex polarization

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[76] The unit vector $\mathcal{N}^\mu(\tau,\vec{\sigma})$ contains a DO dependence in the overall normalizing factor. The existence of this space-like gauge direction seems to indicate that *synchronous or time orthogonal* 4-coordinates with $N_r(\tau,\vec{\sigma}) = -{}^4g_{\tau r}(\tau,\vec{\sigma}) = 0$ (absence of gravito-magnetism) have singular nature [115]. Note that the evolution vector of the *slicing point of view* has $N(\tau,\vec{\sigma})\, l^\mu(\tau,\vec{\sigma})$ and $|\vec{N}(\tau,\vec{\sigma})|\mathcal{N}^\mu(\tau,\vec{\sigma})$ as projections along the normal and the plane tangent to Σ_τ, respectively.

[77] This requires that $\mathcal{N}_\mu\, dx^\mu$ is a closed 1-form, namely that in adapted coordinates we have $\partial_\tau\, \frac{N_r}{|\vec{N}|} = \partial_r\, |\vec{N}|$ and $\partial_r\, \frac{N_s}{|\vec{N}|} = \partial_s\, \frac{N_r}{|\vec{N}|}$. This requires in turn $\frac{N_r}{|\vec{N}|} = \partial_r\, f$ with $\partial_\tau\, f = |\vec{N}| + const.$

vectors $E^{\mu}_{\vec{\sigma}_o\,(\pm)}(\tau)$, like in electromagnetism. They are built starting from the transverse helicity polarization vectors $E^{\mu}_{\vec{\sigma}_o\,(1,2)}(\tau)$, which are the first and second columns of the standard Wigner helicity boost generating $\mathcal{K}^{\mu}_{\vec{\sigma}_o}(\tau)$ from the reference vector $\overset{\circ}{\mathcal{K}}{}^{\mu}_{\vec{\sigma}_o}(\tau) = |\vec{N}|\,\Big(1;001\Big)$ (see for instance the Appendices of Ref. [117]).

Let us call $E^{(ADM)\mu}_{\vec{\sigma}_o\,(\alpha)}(\tau)$ the ADM tetrad formed by $l^{\mu}_{\vec{\sigma}_o}(\tau)$, $\mathcal{N}^{\mu}_{\vec{\sigma}_o}(\tau)$, $E^{\mu}_{\vec{\sigma}_o\,(1,2)}(\tau)$ [78]. This tetrad will not be in general Fermi-Walker transported along the world-line $x^{\mu}_{\vec{\sigma}_o}(\tau)$ of the observer[79].

Another possible (but only on-shell) choice of the spatial triad together with the unit normal to Σ_τ is the *local WSW (on-shell) compass of inertia*, namely the triads transported with the Frauendiener-Sen-Witten transport (see footnote 73 and Eq.(12.2) of Ref. [44]) starting from an asymptotic conventional triad (choice of the fixed stars) added to the ADM 4-momentum at spatial infinity. As shown in Eq.(12.3) of Ref. [44], they have the expression $E^{(WSW)\mu}_{\vec{\sigma}_o(a)}(\tau) = \frac{\partial z^{\mu}_{\Sigma}}{\partial \sigma^s}|_{x_{\vec{\sigma}_o}(\tau)}\, {}^3e^{(WSW)s}_{\vec{\sigma}_o\,(a)}(\tau)$ where the triad ${}^3e^{(WSW)}_{\vec{\sigma}_o(a)}$ is solution of the Frauendiener-Sen-Witten equation restricted to a solution of Einstein equations.

Given an observer with world-line $x^{\mu}_{\vec{\sigma}_o}(\tau)$ and tetrad $E^{\mu}_{\vec{\sigma}_o\,(\alpha)}(\tau)$, the geometrical properties are described by the Frenet-Serret equations [118]

$$\begin{aligned}
&\frac{D}{D\tau}\, l^{\mu}_{\vec{\sigma}_o}(\tau) = \kappa_{\vec{\sigma}_o}(\tau)\, E^{\mu}_{\vec{\sigma}_o\,(1)}(\tau),\\
&\frac{D}{D\tau}\, E^{\mu}_{\vec{\sigma}_o\,(1)}(\tau) = a^{\mu}_{\vec{\sigma}_o}(\tau) = \kappa_{\vec{\sigma}_o}(\tau)\, l^{|mu}_{\vec{\sigma}_o}(\tau) + \tau_{\vec{\sigma}_o\,(1)}(\tau)\, E^{\mu}_{\vec{\sigma}_o\,(2)}(\tau),\\
&\frac{D}{D\tau}\, E^{\mu}_{\vec{\sigma}_o\,(2)}(\tau) = -\tau_{\vec{\sigma}_o\,(1)}(\tau)\, E^{\mu}_{\vec{\sigma}_o\,(1)}(\tau) + \tau_{\vec{\sigma}_o\,(2)}(\tau)\, E^{\mu}_{\vec{\sigma}_o\,(3)}(\tau),
\end{aligned}$$

---

[78] It is a tetrad in adapted coordinates: if $E^{\mu}_{(\alpha)} = \frac{\partial z^{\mu}_{\Sigma}}{\partial \sigma^A}\, E^A_{(\alpha)}$, then $E^{(ADM)\,A}_{\vec{\sigma}_o\,(\alpha)}(\tau)\, {}^4g_{AB}(\tau, \vec{\rho}_{\vec{\sigma}_o}(\tau))\, E^{(ADM)\,B}_{\vec{\sigma}_o\,(\beta)}(\tau) = {}^4\eta_{(\alpha)(\beta)}$.

[79] Given the 4-velocity $l^{\mu}_{\vec{\sigma}_o}(\tau) = E^{\mu}_{\vec{\sigma}_o}(\tau)$ of the observer, the spatial triads $E^{\mu}_{\vec{\sigma}_o\,(a)}(\tau)$, $a = 1,2,3$, have to be chosen in a conventional way, namely by means of a conventional assignement of an origin for the local measurements of rotations. Usually, the choice corresponds to Fermi-Walker (FW) transported (*gyroscope-type transport, non-rotating observer*) tetrads $E^{(FW)\,\mu}_{\vec{\sigma}_o\,(\alpha)}(\tau)$, such that

$$\begin{aligned}
\frac{D}{D\tau}\, E^{(FW)\,\mu}_{\vec{\sigma}_o\,(a)}(\tau) &= \Omega^{(FW)\mu}_{\vec{\sigma}_o}{}_{\nu}(\tau)\, E^{(FW)\,\nu}_{\vec{\sigma}_o\,(a)}(\tau) = l^{\mu}_{\vec{\sigma}_o}(\tau)\, a_{\vec{\sigma}_o\,\nu}(\tau)\, E^{(FW)\,\nu}_{\vec{\sigma}_o\,(a)}(\tau),\\
\Omega^{(FW)\mu\nu}_{\vec{\sigma}_o}(\tau) &= a^{\mu}_{\vec{\sigma}_o}(\tau)\, l^{\nu}_{\vec{\sigma}_o}(\tau) - a^{\nu}_{\vec{\sigma}_o}(\tau)\, l^{\mu}_{\vec{\sigma}_o}(\tau).
\end{aligned}$$

The triad $E^{(FW)\,\mu}_{\vec{\sigma}_o\,(a)}(\tau)$ is the correct relativistic generalization of *global Galilean non-rotating frames* (see Ref. [82]) and is defined using only local geometrical and group-theoretical concepts. Any other choice of the triads (Lie transport, co-rotating-FW transport,...) is obviously also possible [86]. A generic triad $E^{\mu}_{\vec{\sigma}_o\,(a)}(\tau)$ will satisfy $\frac{D}{D\tau}\, E^{\mu}_{\vec{\sigma}_o\,(a)}(\tau) = \Omega_{\vec{\sigma}_o}{}^{\mu}{}_{\nu}(\tau)\, E^{\nu}_{\vec{\sigma}_o\,(a)}(\tau)$ with $\Omega^{\mu\nu}_{\vec{\sigma}_o} = \Omega^{(FW)\,\mu\nu}_{\vec{\sigma}_o} + \Omega^{(SR)\,\mu\nu}_{\vec{\sigma}_o}$ with the spatial rotation part $\Omega^{(SR)\,\mu\nu}_{\vec{\sigma}_o} = \epsilon^{\mu\nu\alpha\beta}\, l_{\vec{\sigma}_o\,\alpha}\, J_{\vec{\sigma}_o\,\beta}$, $J^{\mu}_{\vec{\sigma}_o}\, l_{\vec{\sigma}_o\,\mu} = 0$, producing a rotation of the gyroscope in the local space-like 2-plane orthogonal to $l^{\mu}_{\vec{\sigma}_o}$ and $J^{\mu}_{\vec{\sigma}_o}$.

$$\frac{D}{D\tau}\, E^{\mu}_{\vec{\sigma}_o\,(3)}(\tau) = -\tau_{\vec{\sigma}_o\,(2)}(\tau)\, E^{\mu}_{\vec{\sigma}_o\,(2)}(\tau), \tag{B6}$$

where $\kappa_{\vec{\sigma}_o}(\tau)$, $\tau_{\vec{\sigma}_o(a)}(\tau)$, $a = 1, 2$, are the curvature and the first and second torsion of the world-line. $E^{\mu}_{\vec{\sigma}_o\,(a)}(\tau)$, $a = 1, 2, 3$ are said the normal and the first and second bi-normal of the world-line, respectively.

Let us now look at the description of a geodesics $y^{\mu}(\tau)$, the world-line of a scalar test particle, from the point of view of those observers $\gamma_{\vec{\sigma}_o, y(\tau)}$ of the congruence who intersect it, namely such that at τ it holds $x^{\mu}_{\vec{\sigma}_o, y(\tau)}(\tau) = y^{\mu}(\tau)$. The family of these observers is called a *relative observer world 2-sheet* in Ref. [86].

Since the parameter τ labeling the leaves Σ_{τ} of the foliation is not the proper time $s = s(\tau)$ of the geodesics $y^{\mu}(\tau) = Y^{\mu}(s(\tau))$, the geodesics equation $\frac{d^2 Y^{\mu}(s)}{ds^2} + {}^4\Gamma^{\mu}_{\alpha\beta}(Y(s))\, \frac{dY^{\alpha}(s)}{ds}\, \frac{dY^{\beta}(s)}{ds} = 0$ (or $m\, a^{\mu}(s) = m\, \frac{d^2 Y^{\mu}(s)}{ds^2} = F^{\mu}(s)$, where m is the mass of the test particle), becomes

$$\frac{d^2 y^{\mu}(\tau)}{d\tau^2} + {}^4\Gamma^{\mu}_{\alpha\beta}(y(\tau))\, \frac{dy^{\alpha}(\tau)}{d\tau}\, \frac{dy^{\beta}(\tau)}{d\tau} - \frac{dy^{\mu}(\tau)}{d\tau}\, \frac{d^2 s(\tau)}{d\tau^2}\, \Big(\frac{ds(\tau)}{d\tau}\Big)^{-1} = 0, \tag{B7}$$

or

$$m\, a^{\mu}_y(\tau) = m\, \frac{d^2 y^{\mu}(\tau)}{d\tau^2} = f^{\mu}(\tau). \tag{B8}$$

We see that the force $f^{\mu}(\tau)$ contains an extra-piece with respect to $F^{\mu}(s(\tau))$, due to the change of time parameter.

Let $U^{\mu}(\tau) = V^{\mu}(s(\tau)) = \frac{dY^{\mu}(s)}{ds}|_{s=s(\tau)} = \frac{\dot{y}^{\mu}(\tau)}{\sqrt{{}^4g_{\alpha\beta}(y(\tau))\, \dot{y}^{\alpha}(\tau)\, \dot{y}^{\beta}(\tau)}}$ with $\dot{y}^{\mu}(\tau) = \frac{dy^{\mu}(\tau)}{d\tau}$ be the 4-velocity of the test particle and $ds = \sqrt{{}^4g_{\alpha\beta}(y(\tau))\, \dot{y}^{\alpha}(\tau)\, \dot{y}^{\beta}(\tau)}\, d\tau$ be the relation between the two parameters. By using the *intrinsic or absolute derivative* along the geodesics parametrized with the proper time $s = s(\tau)$, the geodesics equation becomes $\mathcal{A}^{\mu}(s) = \frac{DV^{\mu}(s)}{ds} = 0$ [or $\tilde{\mathcal{A}}^{\mu}(\tau) = \frac{DU^{\mu}(\tau)}{d\tau} = \frac{dy^{\mu}(\tau)}{d\tau}\, \frac{d^2 s(\tau)}{d\tau^2}\, \Big(\frac{ds(\tau)}{d\tau}\Big)^{-1} = g^{\mu}(\tau)$].

In non-relativistic physics *spatial inertial forces are defined as minus the spatial relative accelerations*, with respect to an accelerated *global Galilean frame* (see Ref. [82]). In general relativity one needs the whole relative observer world 2-sheet to define an abstract 3-path in the quotient space of space-time by the observer-family world-lines, representing the trajectory of the test particle in the observer *3-space*. Moreover, a well defined projected *time derivative* is needed to define a relative acceleration associated to such 3-path. At each point $P(\tau)$ of the geodesics, identified by a value of τ, we have the two vectors $U^{\mu}(\tau)$ and $l^{\mu}_{\vec{\sigma}_o\, y(\tau)}(\tau)$. Therefore, each vector X^{μ} in the tangent space to space-time in that point $P(\tau)$ admits two splittings:

i) $X^{\mu} = X_U\, U^{\mu} + P(U)^{\mu}{}_{\nu}\, X^{\nu}$, $P^{\mu\nu}(U) = {}^4g^{\mu\nu} - U^{\mu}\, U^{\nu}$, i.e., into a temporal component along U^{μ} and a spatial transverse component, living in the local rest frame LRS_U;

ii) $X^{\mu} = X_l\, l^{\mu}_{\vec{\sigma}_o y(\tau)} + P(l_{\vec{\sigma}_o y(\tau)})^{\mu}{}_{\nu}\, X^{\nu}$, i.e., into a temporal component along $l^{\mu}_{\vec{\sigma}_o y(\tau)}(\tau)$ and a spatial transverse component, living in the local rest frame LRS_l, which is the plane tangent to the leave Σ_{τ} in $P(\tau)$ for our surface-forming congruence.

The *measurement* of X^μ by the observer congruence consists in determining the scalar X_l and the spatial transverse vector. In adapted coordinates and after a choice of the spatial triads, the spatial transverse vector is described by the three (coordinate independent) tetradic components $X_{(a)} = E^\mu_{(a)}\, X_\mu$. The same holds for every tensor. Moreover, every spatial vector like $P(U)^\mu{}_\nu\, X^\nu$ in LRS_U admits a 2+1 orthogonal decomposition (*relative motion orthogonal decomposition*) into a component in the 2-dimensional rest subspace $LRS_U \cap LRS_L$ transverse to the direction of relative motion and one component in the 1-dimensional (longitudinal) orthogonal complement along the direction of the relative motion in each such rest space.

At each point $P(\tau)$, the tangent space is split into the *relative observer 2-plane* spanned by $U^\mu(\tau)$ and $l^\mu_{\vec{\sigma}_o y(\tau)}(\tau)$ and into an orthogonal space-like 2-plane. We have the 1+3 orthogonal decomposition

$$\begin{aligned}
U^\mu(\tau) &= \gamma(U,l)(\tau)\,\Big(l^\mu_{\vec{\sigma}_o\, y(\tau)}(\tau) + \nu^\mu(U,l)(\tau)\Big),\\
&\gamma(U,l) = U_\mu\, l^\mu_{\vec{\sigma}_o y(\tau)}, \qquad \nu(U,l) = \sqrt{\nu^\mu(U,l)\,\nu_\mu(U,l)},\\
&\hat{\nu}^\mu(U,l) = \frac{\nu^\mu(U,l)}{\nu(U,l)}, \qquad relative\;\; 4-velocity\;\; tangent\, to\, \Sigma_\tau.
\end{aligned} \tag{B9}$$

The equation of geodesics, written as $m\, \mathcal{A}^\mu(s) = 0$, is described by the observers' family as:

i) a temporal projection along $l^\mu_{\vec{\sigma}_o y(\tau)}$, leading to the evolution equation $m\, \mathcal{A}_\mu\, l^\mu_{\vec{\sigma}_o y(\tau)} = 0$, for the observed energy ($E(U,l) = \gamma(U,l)$) of the test particle along its world-line;

ii) a spatial projection orthogonal to $l^\mu_{\vec{\sigma}_o y(\tau)}$ (tangent to Σ_τ), leading to the evolution equation for the observed 3-momentum of the test particle along its world-line, with the kinematical quantities describing the motion of the family of observers entering as *inertial forces*. If, instead of writing $m\, P(l)^\mu{}_\nu\, \mathcal{A}^\mu(s) = 0$ with $P(l)^{\mu\nu} = {}^4g^{\mu\nu} - l^\mu_{\vec{\sigma}_o y(\tau)}\, l^\nu_{\vec{\sigma}_o y(\tau)}$, we rescale the particle proper time $s(\tau)$ to the sequence of observer proper times $s_{(U,l)}$ defined by $\frac{ds_{(U,l)}}{ds} = \gamma(U,l)$, the spatial projection of the geodesics equation, re-scaled with the gamma factor [80], can be written in the form

$$\begin{aligned}
m\,\Big(\frac{D_{(FW)}(U,l)}{ds_{(U,l)}}\Big)^\mu{}_\nu\, v^\nu(U,l) &= m\, a^\mu_{(FW)}(U,l) = F^{(G)\mu}_{(FW)}(U,l),\\
F^{(G)\mu}_{(FW)}(U,l) &= -\gamma(U,l)^{-1}\, P^\mu{}_\nu(l)\, \frac{D l^\nu_{\vec{\sigma}_o y(\tau)}(\tau(s))}{ds} =\\
&= -\Big(\frac{D_{(FW)}(U,l)}{ds_{(U,l)}}\Big)^\mu{}_\nu\, l^\nu_{\vec{\sigma}_o y(\tau)}(\tau(s_{(U,l)})) =\\
&= \gamma(U,l)\,\Big[-a^\mu(l) + \Big(-\omega^\mu{}_\nu(l) + \theta^\mu{}_\nu(l)\Big)\nu^\nu(U,l)\Big],
\end{aligned} \tag{B10}$$

where $v^\mu(U,l) = U^\mu - \gamma(U,l)\, l^\mu_{\vec{\sigma}_o y(\tau)} = v(U,l)\, \hat{\nu}^\mu(U,l)$ with $v(U,l) = \gamma(U,l)\, \nu(U,l)$, and $P(l)^\mu{}_\nu\, \frac{D}{ds} = \Big(\frac{D_{(FW)}(U,l)}{ds}\Big)^\mu{}_\nu$ is the spatial FW intrinsic derivative along the test world-line

[80] Namely $m\, \gamma^{-1}(U,l)\, P(l)^\mu{}_\nu\, \mathcal{A}^\nu = 0$.

and $a^{\mu}_{(FW)}(U,l)$ is the *FW relative accelaration.* The term $F^{(G)\mu}_{(FW)}(U,l)$ can be interpreted as the set of *inertial forces* due to the motion of the observers themselves, as in the non-relativistic case. Such inertial forces depend on the following congruence properties:

i) the acceleration vector field $a^{\mu}(l)$, leading to a *gravito-electric field and a spatial gravito-electric gravitational force*;

ii) the vorticity ${\omega^{\mu}}_{\nu}(l)$ and expansion + shear ${\theta^{\mu}}_{\nu}(l) = \sigma^{\mu}_{\nu}(l) + \frac{1}{3}\,\Theta(l)\,{P^{\mu}}_{\nu}(l)$ mixed tensor fields, leading to a *gravito-magnetic vector field and tensor field and a Coriolis or gravito-magnetic force* linear in the relative velocity $\nu^{\mu}(U,l)$.

Then, by writing $v^{\mu}(U,l) = v(U,l)\,\hat{\nu}^{\mu}(U,l)$, the FW relative acceleration can be decomposed into a longitudinal and a transverse relative acceleration

$$
\begin{aligned}
a^{\mu}_{(FW)}(U,l) &= \frac{D_{(FW)}(U,l)\,v(U,l)}{ds_{(U,l)}}\,\hat{\nu}^{\mu}(U,l) + \gamma(U,l)\,a^{(\perp)\mu}_{(FW)}(U,l),\\
a^{(\perp)\mu}_{(FW)}(U,l) &= v(U,l)\,\Big(\frac{D_{(FW)}}{ds_{(U,l)}}\Big)^{\mu}{}_{\nu}\,\hat{\nu}^{\nu}(U,l) =\\
&= \nu^2(U,l)\,\Big(\frac{D_{(FW)}}{dr_{(U,l)}}\Big)^{\mu}{}_{\nu}\,\hat{\nu}^{\nu}(U,l) = \frac{\nu^2(U,l)}{\rho_{(FW)}(U,l)}\,\hat{\eta}^{\mu}_{(FW)}(U,l).
\end{aligned}
\tag{B11}
$$

In the second expression of the transverse FW relative acceleration, the reparametrization $\frac{dr_{(U,l)}}{ds_{(U,l)}} = \nu(U,L)$ to a spatial arclength parameter has been done. Since $\gamma(U,l)\,a^{(\perp)\mu}_{(FW)}(U,l)$ is the transverse part of the relative acceleration, i.e. the *FW relative centripetal acceleration*, $-m\,\gamma(U,l)\,a^{(\perp)\mu}_{(FW)}(U,l)$ may be interpreted as a *centrifugal force*, so that the geodesics equation is rewritten as $m\,\frac{D_{(FW)}(U,l)\,v(U,l)}{ds_{(U,l)}}\,\hat{\nu}^{\mu}(U,l) = F^{(G)\mu}_{(FW)}(U,l) - m\,\gamma(U,l)\,a^{(\perp)\mu}_{(FW)}(U,l)$, with the first member called sometimes *Euler force.*

The 3-path in the abstract quotient space can be treated as an ordinary 3-curve in a 3-dimensional Riemann space. Its tangent is $\hat{\nu}^{\mu}(U,l)$, while its normal and bi-normal are denoted $\hat{\eta}^{\mu}_{(FW)}(U,l)$ and $\hat{\xi}^{\mu}_{(FW)}(U,l)$ respectively. The 3-dimensional Frenet-Serret equations are then

$$
\begin{aligned}
\Big(\frac{D_{(FW)}(U,l)}{dr_{(U,l)}}\Big)^{\mu}{}_{\nu}\,\hat{\nu}^{\nu}(U,l) &= \kappa_{(FW)}(U,l)\,\hat{\eta}^{\mu}_{(FW)}(U,l),\\
\Big(\frac{D_{(FW)}(U,l)}{dr_{(U,l)}}\Big)^{\mu}{}_{\nu}\,\hat{\eta}^{\nu}_{(FW)}(U,l) &= -\kappa_{(FW)}(U,l)\,\hat{\nu}^{\mu}(U,l) + \tau_{(FW)}(U,l)\,\hat{\xi}^{\mu}_{(FW)}(U,l),\\
\Big(\frac{D_{(FW)}(U,l)}{dr_{(U,l)}}\Big)^{\mu}{}_{\nu}\,\hat{\xi}^{\nu}_{(FW)}(U,l) &= -\tau_{(FW)}(U,l)\,\hat{\eta}^{\mu}_{(FW)}(U,l),
\end{aligned}
\tag{B12}
$$

where $\kappa_{(FW)}(U,l) = 1/\rho_{(FW)}(U,l)$ and $\tau_{(FW)}(U,l)$ are the curvature and torsion of the 3-curve, respectively.

The main drawback of the 1+3 (*threading*) description, notwithstanding its naturality from a locally operational point of view, is the use of a rotating congruence of time-like observers: this introduces an element of non-integrability and, as yet, no formulation of the Cauchy problem for the 1+3 reformulation of Einstein's equations has been worked out.

APPENDIX C: AXIOMATIC FOUNDATIONS AND THEORY OF MEASUREMENT IN GENERAL RELATIVITY.

In this Appendix we review the axiomatic approach to the theory of measurement in general relativity by means of idealized test matter.

After a critique of the Synge's *chronometric* axiomatic approach [31] [81],Ehlers, Pirani and Schild (see Ref. [40]), reject *clocks* as basic tools for setting up the space-time geometry and propose to use *light rays* and *freely falling particles*. The full space-time geometry can then be synthesized from a few local assumptions about light propagation and free fall.

a) The propagation of light determines at each point of space-time the infinitesimal null cone and thus establishes its *conformal structure* $\mathcal{C}$. In this way one introduces the notions of being space-like, time-like and null and one can single out as *null geodesics* the null curves contained in a null hyper-surface (the light rays).

b) The motions of freely falling particles determine a family of preferred $\mathcal{C}$-time-like curves. By assuming that this family satisfies a generalized law of inertia (existence of local inertial frames in free fall, equality of inertial and passive gravitational mass), it follows that free fall defines a *projective structure* $\mathcal{P}$ in space-time such that the world lines of freely falling particles are the $\mathcal{C}$-time-like geodesics of $\mathcal{P}$.

c) Since, experimentally, an ordinary particle (positive rest mass), though slower than light, can be made to chase a photon arbitrarily close, the conformal and projective structures of space-time are *compatible*, in the sense that every $\mathcal{C}$-null geodesic is also a $\mathcal{P}$-geodesic. This makes M^4 a *Weyl space* $(M^4, \mathcal{C}, \mathcal{P})$. A Weyl space possesses a unique *affine structure* $\mathcal{A}$ such that $\mathcal{A}$-geodesics coincide with $\mathcal{P}$-geodesics and $\mathcal{C}$-nullity of vectors is preserved under $\mathcal{A}$-parallel displacement. In conclusion, light propagation and free fall define a Weyl

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[81] Synge accepts as basic primitive concepts *particles* and *(standard) clocks*. Then he introduces the 4-metric as the fundamental structure, postulating that whenever x, $x + dx$ are two nearby events contained in the world line or history of a clock, then the separation associated with $(x, x + dx)$ equals the time interval as measured by that (and by other suitably scaled) clock. These axioms are good for the *deduction* of the subsequent theory, but are not a good *constructive* set of axioms for relativistic space-times geometries. The Riemannian line element cannot be derived by clocks alone without the use of light signals. The chronometric determination of the 4-metric components does not compellingly determine the behaviour of freely falling particles and light rays and Synge has to add a further axiom (the geodesic hypothesis). On the basis of this axiom it is then possible (Marzke [99], Kundt-Hoffmann [119]) to construct clocks by means of freely falling particles and light rays (i.e. to give a physical interpretation of the 4-metric in terms of time). Therefore the chronometric axioms appear either as redundant or, if the term *clock* is interpreted as *atomic clock*, as a link between macroscopic gravitation theory and atomic physics: these authors claim for the equality of gravitational and atomic time. It should be better to test this equality experimentally (in radar tracking of planetary orbits atomic time has been used *only* as an *ordering parameter*, whose relation to gravitational time was to be determined from the observations) or to derive it eventually from a theory that embraces both gravitational and atomic phenomena, rather than to postulate it as an axiom.

structure $(M^4, \mathcal{C}, \mathcal{A})$ on space-time (this is equivalent to an *affine connection* due to the presence of both the projective and the conformal structure).

d) In a Weyl space-time, one can define an *arc length* (unique up to linear transformations) along any non-null curve. Applying such definition to the time-like world line of a particle P (not necessarily freely falling), we obtain a *proper time* (= arc length) t on P, provided two events on P have been selected as *zero point* and *unit point of time*. The (idealized) Kundt-Hoffmann experiment [119] designed to measure proper time along a time-like world line in Riemannian space-time by means of light signals and freely falling particles can be used without modifications to measure the proper time t in a Weyl space-time.

e) In absence of a *second clock effect* [82] a Weyl space $(M^4, \mathcal{C}, \mathcal{A})$ becomes a Riemannian space, in the sense that there exists a Riemannian 4-metric $\mathcal{M}$ compatible with $\mathcal{C}$ (i.e. having the same null-cones) and having $\mathcal{A}$ as its metric connection. The Riemannian metric is necessarily unique up to a constant positive factor. Since $\mathcal{A}$ determines a *curvature tensor* R, the use of the equation of geodesic deviation shows that $(M^4, \mathcal{C}, \mathcal{A})$ is Riemannian if and only if the proper times t, $t^{'}$ of two arbitrary, infinitesimally close, freely falling particles P, $P^{'}$ are *linearly* related (to first order in the distance) by *Einstein simultaneity* (see Ref. [40]). In Newtonian space-time the role of $\mathcal{C}$ is played by the *absolute time*. It is also easy to add a physically meaningful axiom that singles out the space-time of special relativity, either by requiring homogeneity and isotropy of M^4 with respect to $(\mathcal{C}, \mathcal{A})$, or by postulating vanishing relative accelerations between arbitrary, neighboring, freely falling particles.

Now, Perlick [94] states that experimental data on standard atomic clocks confirm the absence of the *second clock effect*, so that our actual space-time is not Weyl but pseudo-Riemannian and it is possible to introduce a notion of *rigid rod*.

Let us note that the previous axiomatic approach should be enlarged to cover tetrad gravity, because of the need of test gyroscopes to define the triads of the tetrads of time-like observers. Then the axiomatics would include the possibility of measuring gravito-magnetism and would have to face the question of whether or not the free fall of macroscopic test gyroscopes is geodesic.

An associated theory of the measurement of time-like and space-like intervals has been developed by Martzke-Wheeler [88,99], using Schild geodesic clock (if it is a standard clock, Perlick's definition of rigid rod can be used): *the axiomatics is replaced by the empirical notion of a fiducial interval as standard.* Pauri and Vallisneri [120] have further developed the Martzke-Wheeler approach, showing that, given the *whole* world-line of an accelerated time-like observer, it is possible to build an associated space-time foliation with simultaneity space-like non-overlapping 3-surfaces. This is to be contrasted with the local construction of Fermi coordinate systems: it requires only a local knowledge of the observer world-line but its validity is limited to a neighborhood of the observer, determined by the acceleration radii,

As already said, material (test) reference fluids were introduced by various authors [90,**?**,93] for simulating the axioms.

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[82] The first clock effect is essentially the twin paradox effect. On the other hand, if the time unit cannot be fixed for all standard clocks simultaneously in a consistent way, Perlick [94] speaks of a *second clock effect*

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